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Invariant Groups on Equivalent Crystallizations†

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For each n-crystallization H, $n \geq 2$, we associate two sequences of groups $\xi_k^n(H)$, $0 \leq k \leq n-1$. These groups are proved to be invariant under the crystallization moves [4]. Therefore they are topological invariants for the closed PL n-manifolds. From the second term on each member of each sequence is a quotient group of its predecessor; also each $\dot{\xi}_k^n(H)$ is a quotient of $\xi_k^n(H)$. By the main result of [8], $\xi_{n-1}^n(H)$ (the smallest group) is the fundamental group of $|K(H)|$.

1. INTRODUCTION

An $(n+1)$-graph ($r \geq 2$) is a graph which is finite, regular of valence $n+1$ and properly edge-colored (i.e. at each vertex no two edges have the same color) with $n+1$ colors. For the basic concepts from graph theory we refer to [1]. There exists a standard construction which associated to an $(n+1)$-graph G a pseudo-complex [9, p. 49] $K(G)$ such that $|K(G)|$ is a pseudo-manifold of dimension n. See the preliminares of [5] for this construction. Let $\Delta_n = \{0, 1, \ldots, n\}$ be the set of colors of the edges of an $(n+1)$-graph G. For $\mathscr{B}$ a subset of Δ_n denote by $G_{\mathscr{B}}$ the subgraph of G induced by all the edges with colors in $\mathscr{B}$ and by $\hat{\mathscr{B}}$ the complement of $\mathscr{B}$ in Δ_n. A $\mathscr{B}$-residue [17] is a connected component of G_B. A $\mathscr{B}$-residue, where $\mathscr{B}$ has k colors, is also called a k-residue. A 2-residue is also called a *bigon* [11] (*bi*colored poly*gon*). A bigon in color i and j is specified as an *(i, j)-gon*. If for each i every n-residue $R_{\hat{i}}$ of $G_{\hat{i}}$ is such that $|K(R_{\hat{i}})|$ is homeomorphic to the $(n-1)$-sphere, then the pseudo-manifold $|K(G)|$ is a closed n-manifold. In this case G is called a *gem* (*g*raph-*e*ncoded *m*anifold). For $n = 3$ there are simple arithmetical conditions which characterize when a 4-graph is a gem [7, 13]. For $n = 2$ there are no restrictions: a 3-graph always induces a closed surface. For $n = 4$, 5 this characterization is an open problem. For $n \geq 6$ the algorithmic problem associated with this characterization is undecidable [18]. An important fact about gems is that every closed PL n-manifold M^n is *represented* by a gem in the sense that $M^n \simeq |K(G)|$ for some gem G. An $(n+1)$-graph is called an *n-crystallization* [2, 16] if G is a gem and every $G_{\hat{i}}$, $i \in \Delta_n$, is connected. Every closed PL n-manifold is represented by a crystallization. See [15], [2] or [13] for a direct construction (made for $n = 3$ but which easily generalizes).

Crystallization theory has recently provided a beautiful characterization (non-algorithmic) for the n-sphere [6]. Moreover, it yields a completely graph-theoretical counterpart for homeomorphisms between closed PL n-manifolds [4]. This result has been considerably sharpened by the Switching Lemma proved in [3]. The topological invariance of our groups relies on the basic results of the last two articles, which are summarized in the next section.

2. BASIC RESULTS ON CRYSTALLIZATIONS

Suppose that y is an edge of an $(n+1)$-graph G. Denote by $\mathscr{B}_y$ the set of colors of all the edges with the same ends as y. The edge y is *separating* if its ends belong to distinct $\hat{\mathscr{B}}$-residues. The edge y is *separating* if its ends belong to distinct $\hat{\mathscr{B}}_y$-residues. The *fusion* [13] of y is the process of deletion of the ends of y together with all the edges

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linking them followed by the welding of the free ends along edges of the same color. The resulting $(n+1)$-graph is denoted $G(\text{fus})y$. If G is a gem and y is separating, then $|K(G)| \simeq |K(G(\text{fus})y)|$ [4]. In this case the set of h edges with the same two ends as y (y included) is called an *h-dipole* and the fusion of y or any parallel to it is a *cancellation* of the h-dipole. The inverse operation is named a *creation* of h-dipole. The colors in $\mathcal{B}_y$ are said to be *involved* [3] in the dipole cancellation or creation. The following two modifications of an n-crystallization yielding another are called *crystallization moves*:

(i) creation of a 1-dipole followed by the cancellation of another 1-dipole;
(ii) creation or cancellation of an h-dipole, $1<h<n$.

EQUIVALENCE THEOREM [4]. *Let H and J be n-crystallizations. The n-manifolds $|K(H)|$ and $/K(J)|$ are homeomorphic iff J is obtainable from H by a finite sequence of crystallization moves.*

Crystallizations representing the same manifold are equivalent crystallizations:

SWITCHING LEMMA [3]. *Let H_{ij} denote the crystallization obtained from an n-crystallization H by exchanging two distinct arbitrary colors i and j. Then H_{ij} is obtainable from H by a finite sequence of crystallization moves, each one of these involving color i and not involving color j.*

To prove the invariance of our groups we need (i) of the following corollary of the Switching Lemma:

STRONG EQUIVALENCE THEOREM [3]. *The n-crystallizations H and J are equivalent iff they are equivalent under crystallizations moves that (for the whole sequence of moves) either*

(i) always involve a fixed color i, or
(ii) never involve color i.

3. GROUP ξ^n, $(n+1)^*$-GRAPHS AND GROUPS ξ^n_k

Let H be an n-crystallization, $n \geq 2$. We construct a group $\xi^n(H)$ as follows. For groups given by generators and relations we refer to [14] or [10]. The generators of $\xi^n(H)$ form an abstract set of symbols in one-to-one correspondence with the vertices of H. This correspondence enables us to identify the set of generators with the set of vertices $V(H)$. The set of relators of $\xi^n(H)$ is in correspondence with the set of bigons of H. A relator b_{ij} is constructed for each (i,j)-gon as follows. Let $v_1, v_2, \ldots, v_{2m-1}, v_{2m}$ be the sequence of vertices around the (i,j)-gon with arbitrary starting point and sense of traversal. Take b_{ij} to be $v_1v_2^{-1}\cdots v_{2m-1}v_{2m}^{-1}$ if the vertices with odd indices are the ones which make transitions from the greater to the smaller color with respect to the chosen sense of traversal. If those vertices are the even indexed ones, take b_{ij} to be $v_1^{-1}v_2\cdots v_{2m-1}^{-1}v_{2m}$. Let $b_{ij}(H)$ be the set of all the relators coming this way from the (i,j)-gons. Define also $B(H)$ as $\bigcup\{B_{ij}(H) \mid 0 \leq i<j \leq n\}$. The group $\xi^n(H)$ has, by definition, the presentation

$$\xi^n(H) = [V(H) \mid B(H)].$$

THEOREM 1. *The isomorphism class of the group $\xi^n(H)$ is invariant under the crystallization moves. Therefore $\xi^n(H)$ is a topological invariant for the closed n-manifold $|K(H)|$.*

To obtain the proof of this theorem we need to work with an adequate extension of the class of crystallizations. In Corollary 1 of the last section we prove a result which is more general than this theorem.

An $(n+1)^*$*-graph* is an $(n+1)$-graph which has at most two $\hat{0}$-residues, exactly one $\hat{\imath}$-residue for $i \neq 0$, and is such that each $\{i, j, k\}$-residue, 0 not in $\{i, j, k\}$, represents the 2-sphere. In the terminology of [11] we say that each one of these 3-residues *faithfully* embeds into a 2-sphere (the boundary of the faces are bigons). In fact the faithful embedding of a 3-graph G is the dual of $K(G)$. A 3-residue representing the 2-sphere is said to be *spherical.* The class of $(n+1)^*$-graphs contains the crystallizations and is adequate for our purposes.

Let H be an $(n+1)^*$-graph and let $V(H)$ be as before. Let k be a number between 0 and $n-1$. By $\mathcal{K}$ we mean the subset of Δ_n defined as the empty set, if $k=0$, or by $\{1, \ldots, k\}$ otherwise. Let $R_{\mathcal{K}}(H)$ be a set of symbols representing the $\mathcal{K}$-residues of H. Let r_v be the term in $R_{\mathcal{K}}(H)$ which corresponds to the $\mathcal{K}$-residue containing vertex v. We are going to adjoint $R_{\mathcal{K}}(H)$ to $V(H)$ as a set of generators to define groups generalizing $\xi^n(H)$. The set of relators $P_{\mathcal{K}}(H)$ in the symbols $V(H) \cup R_{\mathcal{K}}(H)$ is $\{vr_v^{-1} \mid v \in V(H)\}$. If H has a unique $\hat{0}$-residue, we define, for $0 \leqslant k < n$.

$$\xi^n_k(H) = [V(H) \cup R_{\mathcal{K}}(H) \mid B(H) \cup P_{\mathcal{K}}(H)].$$

Note that for $k=0$, $\xi^n_k(H) = \xi^n(H) = V(H)$ since $R_0(H) = V(H)$ and $P_0(H)$ reduce to trivial identities. If H has two $\hat{0}$-residues, let $U(H) \cup W(H)$ be the bipartition of $V(H)$ induced by these residues. Choose an arbitrary u in $U(H)$ and an arbitrary W in $W(H)$. Define, for $0 \leqslant k < n$,

$$\xi^n_k(H) = [V(H) \cup R_{\mathcal{K}}(H) \mid B(H) \cup P_{\mathcal{K}}(H) \cup \{uw^{-1}\}].$$

The chosen symbols u and w are called *connectors.*

4. Independence of Connectors

The definition of $\xi^n_k(H)$ has various arbitrary choices. Clearly, the choice of the starting vertex to form a relator in $B(H)$ is irrelevant, by the cyclic nature of relators and by the imposition to adjust the sign in the exponents with the sense of traversal. The groups seem to depend strongly on the ordering of the color (this is needed). However, for crystallizations (see Corollary 2 in Section 7), this is not the case. Finally, when H has two $\hat{0}$-residues, two arbitrary symbols, the connectors, enter the definition of $\xi^n_k(H)$. Below we prove a lemma which shows that any choice of connectors produces the same group. One more comment: by the presence of $P_{\mathcal{K}}(H)$ the real generators of groups are the $\mathcal{K}$-residues (via its symbols). However, we keep the larger presentation because it is adequate for the proof of Theorem 2 in Section 6.

Lemma 1. *The isomorphism class of $\xi^n_k(H)$, when H has two $\hat{0}$-residues, is independent of the choice of connectors.*

Proof. To simplify the notation we drop the argument H from $V(H)$, $R_{\mathcal{K}}(H)$, $U(H)$, etc. We prove the lemma by showing that we can replace u by any other member, say t, of U. Let R_U (resp. R_W) be the subset of $R_{\mathcal{K}}$ formed by the $\mathcal{K}$-residue with vertices contained in U (resp. W). (From now on we may confound a $\mathcal{K}$-residue with its symbol, as we have been doing for the vertices.) Since a $\mathcal{K}$-residue does not contain 0-colored edges, any member of $R_{\mathcal{K}}$ has a vertex-set entirely contained in R_U or in R_w. Let $S_U = U \cup R_U$ and $S_W = W \cup R_W$. Also define S to be $S_U \cup S_W = V \cup R_{\mathcal{K}}$. Let $s \to \hat{s}$ be a one-to-one correspondence between S and a disjoint new set of symbols $\hat{S}$.

We now use Tietze transformations [10] to modify the presentation of $\xi_k^n(H)$.

$$\xi_k^n(H) = [S \mid B \cup P_{\mathcal{K}} \cup \{uw^{-1}\}] = [\hat{S} \cup S \mid B \cup P_{\mathcal{K}} \cup \{uw^{-1}\} \cup \{\hat{x}x^{-1}tw^{-1} \mid x \in S_U\} \cup \{\hat{y}y^{-1} \mid y \in S_W\}].$$

The second equality holds because each new symbol appears alone in a unique new relator. Next we want to show that the set of relators $\hat{B}$ and $\hat{P}_{\mathcal{K}}$, obtained from B and P_k by replacing each s by $\hat{s}$, are equivalent to these and so can take their place in the above presentation. Let $\prod_{j=1}^{m} x_{2j-1}x_{2j}^{-1}$ be an arbitrary relator in $B \cup P_{\mathcal{K}}$. Let $K(x)$ be either wt^{-1} or the identity in the group $\xi_k^n(H)$, depending on whether or not $x \in S_U$. By using the relations $x_i = [k(x_i)]^{-1}\hat{x}_i$ (equivalent to the new relators), we may rewrite the above relator as

$$[k(x_1)]^{-1}\hat{x}_1\left\{\prod_{j=1}^{m-1} \hat{x}_{2j}^{-1}k(x_{2j})[k(x_{2j+1})]^{-1}\hat{x}_{2j+1}\right\}\hat{x}_{2m}^{-1}k(x_{2m}).$$

If $m = 1$ the product between curly brackets is to be considered the identity. This is the case when the original relator is in $P_{\mathcal{K}}$. Note that for relators in this set $k(x_1) = k(x_2) = k(x_{2m})$. Therefore each such relator is equivalent to $\hat{x}_1\hat{x}_2^{-1}$ and $\hat{P}_{\mathcal{K}} \equiv P_{\mathcal{K}}$. In the case that the original relator is in B observe that the vertices x_{2j} and x_{2j+1} are linked by an edge which is never 0-colored (0 is the smallest color). (See Remark 1 after the proof.) Therefore these vertices are in the same $\hat{0}$-residue and $k(x_{2j}) = k(x_{2j+1})$. The preceding equality holds mod $2m$, namely $k(x_{2m}) = k(x_1)$. Therefore the original relator is in every case equivalent to $\prod_{j=1}^{m-1} \hat{x}_{2j-1}\hat{x}_{2j}^{-1}$ and so $\hat{B} \equiv B$. By using $\hat{w}w^{-1}$ to eliminate w we obtain, where $S^w = S - \{w\}$ and $S_w^w = S_w - \{w\}$, the following presentation:

$$\xi_k^n(H) = [\hat{S} \cup S^w \mid \hat{B} \cup \hat{P}_{\mathcal{K}} \cup \{u\hat{w}^{-1}\} \cup \{\hat{x}x^{-1}t\hat{w}^{-1} \mid x \in S_U\} \cup \{\hat{y}y^{-1} \mid y \in S_W^w\}].$$

Now use $u\hat{w}^{-1}$ to eliminate u. With $S^{wu} = S - \{w, u\}$, $S_U^u = S_U - \{u\}$ we obtain

$$\xi_k^n(H) = [\hat{S} \cup S^{wu} \mid \hat{B} \cup \hat{P}_{\mathcal{K}} \cup \{\hat{u}\hat{w}^{-1}t\hat{w}^{-1}\} \cup \{\hat{x}x^{-1}t\hat{w}^{-1} \mid x \in S_U^u\} \cup \{\hat{y}y^{-1} \mid y \in S_W^w\}].$$

Finally use $\hat{u}\hat{w}^{-1}t\hat{w}^{-1}$ to eliminate t, obtaining

$$\xi_k^n(H) = [\hat{S} \cup S^{wut} \mid \hat{B} \cup \hat{P}_{\mathcal{K}} \cup \{\hat{x}x^{-1}\hat{w}\hat{u}^{-1} \mid x \in S_U^{ut}\} \cup \{\hat{t}\hat{w}^{-1}\} \cup \{\hat{y}y^{-1} \mid y \in S_W^w\}].$$

Note that the symbols in $S^{wut} = S - \{w, u, t\}$ appear just once. Eliminating these symbols and the relators where they appear we obtain

$$\xi_k^n(H) = [\hat{S} \mid \hat{B} \cup \hat{P}_{\mathcal{K}} \cup \{\hat{t}\hat{w}^{-1}\}] \cong [S \mid B \cup P_{\mathcal{K}} \cup \{tw^{-1}\}].$$

which establishes the lemma. □

REMARK 1. *The fact that each edge linking x_{2j} to x_{2j+1} is never 0-colored is the unique reason why we need to have a smallest color to establish that $\hat{B} \equiv B$. This is a point that hinders a generalization of the $(n+1)^*$-graphs to include the gems.*

5. Redundancy on Spherical 3-Residues

Given a connected graph M embedded into a 2-sphere we associate a group Γ_M to it as follows. Orient the edges of M arbitrarily. The generators of Γ_M are symbols in bijection with edges of M, and are identified with them. Its relators are in one-to-one correspondence with the vertices of M. To obtain the relator corresponding to a vertex v list once in clockwise order the edges incident to v. A generator receives the exponent -1 if the edge is *a* directed towards v and $+1$ (no exponent) if the edge is directed away from v. This finishes the description of Γ_M. The following proposition is the basis for the results in this section. The proof is very simple and can be found in [12].

PROPOSITION. *Let m be a connected graph embedded into a 2-sphere. Each relator of Γ_M is a consequence of the remaining relators.*

Let i, j, k be distinct colors in Δ_n. Let k be either the maximum or the minimum of $\{i, j, k\}$. Let T be an $\{i, j, k\}$-residue and, for $h, l \in \Delta_n$, let $B^T_{hl}(H)$ be $B_{hl}(H)$ restricted to T. All these conditions refer to an $(n+1)^*$-graph H. Under them we have:

LEMMA 2. *If T has no 0-colored edges, then each relator in $B^T_{ik} \cup B^T_{jk}$ is a consequence of the remaining relators.*

PROOF. Let M^* denote a faithful embedding of T into a 2-sphere. We modify M^* to arrive at an M where we can use the above proposition. In the interior of each face of M^* bounded by an (i, k)-gon (resp. an (j, k)-gon) put a new vertex, calling it an *i-vertex* (resp. a *j-vertex*). For each k-colored edge y with ends u and w link with two lines, the i- and j-vertices, which lie in faces separated by y. One of these lines must cross y near u and the other must cross it near w. We call these two lines u and w. Topologically these linking lines are to be closed intervals, with just the crossing point in common with M^*, and intersecting among themselves just in their ends, the i- and j-vertices. These vertices and their linking lines form a graph embedded in the 2-sphere which we call M. By this construction the edges of M are in one-to-one correspondence with the vertices of M^*, and the vertices of M are in one-to-one correspondence with the (i, k)-gons and the (j, k)-gons of M^*, embedded T. Choose an orientation for the edges of M as follows. At each i-vertex the left and right lines corresponding to a k-edge are respectively out-directed and in-directed. Thus $[V^T \mid B^T_{ik} \cup B^T_{jk}]$, where V^T is $V(H)$ restricted to T, is Γ_M. The lemma follows from the proposition. □

REMARK 2. The reader should observe that the condition $k = \max\{i, j, k\}$ or $k = \min\{i, j, k\}$ is essential in the above proof. This lemma is the place where we take advantage of the total ordering of the colors $1, 2, \ldots, n$. Here is the other point where the generalization mentioned in Remark 1 is also hindered.

We now develop the concepts and notation needed in the next lemma. An edge y of an $(n+1)^*$-graph H is said to be *fusible* if:

(i) y is separating and 0-colored;
(ii) $\mathcal{Y} = \hat{\mathcal{B}}_y$ has at least two colors;

Let u be an end of a fusible edge y in an $(n+1)^*$-graph H. Let b^u_{ij} be the relator in $B(H)$ associated to the (i, j)-gon containing u. Define β_{uy} to be $\{b^u_{ij} \mid i, j \in \mathcal{Y}\}$ and $B_{uy} = \{b_{ij} \mid b_{ij}$ is in the $\mathcal{Y}$-residue containing $u\}$. (Note that B_{uy} contains ij-gons which does not pass through u.) Let b^{ij}_{uy} be equal to $(B_{uy} - \beta_{uy}) \cup \{b^u_{ij}\}$. Let H' be the $(n+1)^*$-graph obtained from H by fusion of y. The analogous set of relators for H' are as follows. For $i, j \in \mathcal{Y}$, let b^y_{ij} be the relator in $B(H')$ corresponding to the (i, j)-gon which contains the i- and j-colored edges welded in the fusion of y. Let β_y be the set $\{b^y_{ij} \mid i, j \in \mathcal{Y}\}$ and B_y be the subset of $B(H')$ formed by the relators corresponding to bigons in the $\mathcal{Y}$-residue which contains all the edges which were welded. Define b^{ij}_y to be $(B_y - \beta_y) \cup \{b^y_{ij}\}$.

LEMMA 3. *Let y be a fusible edge in an $(n+1)^*$-graph H and let u be one of its ends. Let H' denote H(fus)y and $i < j$ two colors in $\mathcal{Y}$. We have:*

(*a*) *every relator in B_{uy} is implied by the ones in B^{ij}_{uy};*
(*b*) *every relator in B_y is implied by the ones in B^{ij}_y.*

Proof. The proofs of (a) and (b) are given in parallel. Let $k \in \mathcal{Y}$ be distinct from i and j, and let T denote the $\{i, j, k\}$-residue containing the vertex u (in H). Also let T' be the $\{i, j, k\}$-residue of H' which contains the edges colored i, j, k which were welded. Suppose that $k < i$. By Lemma 2, b^u_{ij} is implied by $B^T_{kj} \cup B^T_{ij} - \{b^u_{ij}\}$ and b^y_{kj} is implied by $B^{T'}_{kj} \cup B^{T'}_{ij} - \{b^y_{ij}\}$. Also by Lemma 2, $\{b^u_{ki}\}$ is implied by $B^T_{ki} \cup B^T_{kj} - \{b^u_{ki}\}$ and b^y_{ki} is implied by $B^{T'}_{ki} \cup B^{T'}_{kj} - \{b^y_{ki}\}$. Thus B^{ij}_{uy} implies b^u_{ki} and B^{ij}_y implies b^y_{ij}. Now suppose that $k > i$. By Lemma 2, b^u_{ik} is a consequence of $B^T_{ij} \cup B^T_{ik} - \{b^u_{ik}\}$ and b^y_{ik} is a consequence of $B^{T'}_{ij} \cup B^{T'}_{ik} - \{b^y_{ik}\}$. For every k distinct from i and j we have found that b^u_{ik} is implied by B^{ij}_{uj} and b^y_{ik} is implied by B^{ij}_y. Therefore, if h is a fourth member of $\mathcal{Y}$, b^u_{hk} is implied by $B^{ik}_{uy} \subseteq B^{ij}_{uy} \cup \{b^u_{ik}\}$, and b^y_{hk} is a consequence of $B^{ik}_y \subseteq B^{ij}_y \cup \{b^y_{ik}\}$: use (i, k, h) in place of (i, j, k). For arbitrary colors h, k in $\mathcal{Y}$ we have established that b^u_{hk} follows from B^{ij}_{uy} and that b^y_{hk} from B^{ij}_y. This concludes the proofs of (a) and (b). □

6. Invariance and Fusible Edges

In this section the central result, namely Theorem 2, is proved. Initially we obtain convenient presentations for the groups $\xi^n_k(H)$ and $\xi^n_k(H')$. Let y be a fusible edge in an $(n+1)^*$-graph H and let H' be $H(\text{fus})y$. Let $B_{\hat{y}}$ denote the set of relators in $B(H)$ which correspond to bigons with color in $\hat{\mathcal{Y}} = B_y$ not containing ends of y. Observe that $B_{\hat{y}}$ is also a subset of relators in $B(H')$. Denote by M the subset of relators in $B(H)$, the corresponding bigons of which have one color in $\mathcal{Y}$ and one color in $\hat{\mathcal{Y}}$. Let M' be the set of relators obtained from M by removing from its relators each adjacent occurrence of u and w (the ends of y), the exponents of which add to zero. Note that M' is also a subset of $B(H')$. Relators $B(H)$ and $B(H')$ are partitioned as

$$(*) \qquad B(H) = M \cup B_{\hat{y}} \cup B_{uy} \cup B_{wy} \cup X, \qquad B(H) = M' \cup B_{\hat{y}} \cup B_y,$$

where B_{uy}, B_{wy}, B_y are as in Lemma 3, and X is either the empty set, in the case that $\hat{\mathcal{Y}} = \{0\}$, or $X = \{uw^{-1}\}$ otherwise. Let $P_{\mathcal{K}}(H)$ be partitioned into $\{ur_u^{-1}, wr_w^{-1}\}$, $P_u = \{xr_u^{-1} \neq P_{\mathcal{K}} \mid x \neq u, x \neq w\}$, $P_w = \{xr_w^{-1} \neq P_{\mathcal{K}} \mid x \neq u, x \neq w\}$, and the remaining relators which we denote by $Q_{\mathcal{K}}$. Thus we have

$$(**) \qquad P_{\mathcal{K}}(H) = \{ur_u^{-1}, wr_w^{-1}\} \cup P_u \cup P_w \cup Q_{\mathcal{K}}, \qquad P_{\mathcal{K}}(H') = P_{uw} \cup Q_k,$$

where P_{uw} is $P_u \cup P_w$ with every symbol r_w replaced by r_u. This replacement is in accordance with the case that u and w belong to distinct $\mathcal{K}$-residues of H which became the same of H'. The partition holds in every case: if $\mathcal{K} \subseteq \hat{\mathcal{Y}}$ then $P_u = P_w = P_{uw} = \emptyset$. If $\mathcal{K} \cap \mathcal{Y} \neq \emptyset$, then the $\mathcal{K}$-residues of u and w survive as a common $\mathcal{K}$-residue in H'. If they are the same in H, then $P_{uw} = P_u \cup P_w$, and there is no need of the replacement. If they are not the same the replacement is necessary. Clearly, $Q_{\mathcal{K}}$ stays the same and $\{ur_u^{-1}, wr_w^{-1}\}$ always disappears.

Let B^u_{ij} be as in Lemma 3. Define $b^{\hat{u}}_{ij}$ and $b^{\hat{w}}_{ij}$ to be words in symbols $V(H)$ so that $b^u_{uj} = ub^{\hat{u}}_{ij}$ and $b^w_{ij} = b^{\hat{w}}_{ij}w^{-1}$. (We might have to invert b^u_{ij} and/or b^w_{ij} and negate exponents to obtain the appropriate forms for the definitions of $b^{\hat{u}}_{ij}$ and $b^{\hat{w}}_{ij}$.) Note that $b^{\hat{u}}_{ij}b^{\hat{w}}_{ij}$ is a relator in $B(H')$ and that the following equality is verified:

$$(***) \qquad B^{ij}_{uy} \cup B^{ij}_{wy} \cup \{b^{\hat{u}}_{ij}b^{\hat{w}}_{ij}\} = B^{ij}_y \cup \{ub^{\hat{u}}_{ij}, b^{\hat{w}}_{ij}w^{-1}\}.$$

We impose the condition that the two colors i and j above must be chosen in $\mathcal{Y}$. Since y is fusible this set has at least two colors. Moreover, if $\mathcal{K}$ has a non-empty intersection with $\mathcal{Y}$, i must be chosen in this intersection. Let S be $V(H) \cup R_{\mathcal{K}}(H)$ and let S' be $V(H') \cup R_{\mathcal{K}}(H')$.

LEMMA 4. *The following presentations hold*:

(a) $\xi_k^n(H) = [S \mid M \cup B_{\hat{y}} \cup [B_y^{ij} - \{b_{ij}^{\hat{u}} b_{ij}^{\hat{w}}\}] \cup \{ub_{ij}^{\hat{u}}, b_{ij}^{\hat{w}}w^{-1}, ur_u^{-1}, wr_w^{-1}, uw^{-1}\} \cup P_u \cup P_w \cup Q_{\mathscr{X}}]$;

(b) $\xi_k^n(H') = [S' \mid M' \cup B_{\hat{y}} \cup B_y^{ij}) \cup P_{uw} \cup Q_{\mathscr{X}}]$.

PROOF. Part (a) follows from (***) and from Lemma 3(a) which permits the replacement of the pair B_{uy}, B_{wy} by B_{uy}^{ij}, B_{wy}^{ij} so arriving, except for X, at the partitions of $B(H)$ and $P_{\mathscr{X}}(H)$ given in (*) and (**). We have to take care of X, which might be empty. But in this case H has two $\hat{0}$-residues and we choose u and w as connectors, making sure that in every case $\{uw^{-1}\}$ is a relator. The proof of (b) follows from Lemma 3(b) and from the partitions for $B(H')$ and $P_{\mathscr{X}}(H')$ given in (*) and (**). □

THEOREM 2. *Let y be a fusible edge in an $(n+1)^*$-graph H and $H' = H(\text{fus})y$. Then* $\xi_k^n(H) = \xi_k^n(H')$.

PROOF. The proof uses the presentations of Lemma 4 and is subdivided into two parts according to whether $\mathscr{X} \cap \mathscr{Y}$ is empty or not. Suppose first that it is empty. In this case P_u, P_w and P_{uw} are empty. Use uw^{-1} to eliminate w. Note that w appears, perhaps as r_w, in the three relators where it is explicitly written and in relators of M. This set becomes M' after the elimination of w and the simplifications of the occurrences of uu^{-1}. The current presentation for $\xi_k^n(H)$ is the following:

$$\xi_k^n(H) = [S^w \mid M' \cup B_{\hat{y}} \cup [B_y^{ij} - \{b_{ij}^{\hat{w}} b_{ij}^{\hat{u}}\}] \cup \{ub_{ij}^{\hat{u}} b_{ij}^{\hat{w}} u^{-1}, ur_u^{-1}, ur_u^{-1}, ur_w^{-1}\} \cup Q_{\mathscr{X}}].$$

Observe that now u occurs only in the four relators where it is explicitly written. If $\mathscr{X}$ is empty, the relators ur_u^{-1} and ur_w^{-1} are redundant, since at this point r_w, which is w, has been replaced by u. We then use $ub_{ij}^{\hat{u}}$ to eliminate u and recover $b_{ij}^{\hat{w}} b_{ij}^{\hat{u}}$ as a relator, thus obtaining the presentation of $\xi_k^n(H')$ of Lemma 4(b). If $\mathscr{X}$ is not empty, u and w are in the same $\mathscr{X}$-residue and $r_u \equiv r_w$. Use ur_u^{-1} to eliminate u and then $r_u b_{ij}^{\hat{u}}$ to eliminate r_u. Since u and w are the only vertices of their common $\mathscr{X}$-residue, we arrive at the presentation of Lemma 4(b) once more.

Consider now that i is the color in the intersection of $\mathscr{X}$ and $\mathscr{Y}$. In this case the relators $ub_{ij}^{\hat{u}}$ and $b_{ij}^{\hat{w}} w^{-1}$ are consequences of $P_{\mathscr{X}}(H)$. To see this simply replace each x in these relators by r_x. The symbols which are vertices linked by an i-colored edge have the same r_x and are pairwise cancelled, making the whole relator disappear. Use uw^{-1} to eliminate w and obtain, from Lemma 4(a),

$$\xi_k^n(H) = [S^w \mid M' \cup B_{\hat{y}} \cup [B_y^{ij} - \{b_{ij}^{\hat{w}} b_{ij}^{\hat{u}}\}] \cup \{ur_u^{-1}, ur_w^{-1}\} \cup P_u \cup P_w \cup Q_{\mathscr{X}}].$$

Suppose u and w are in the same $\mathscr{X}$-residue. After the elimination of u via ur_u^{-1} the relator $r_u r_w^{-1}$ is redundant and we obtain, except for the absence of $b_{ij}^{\hat{w}} b_{ij}^{\hat{u}}$, the same presentation of Lemma 4(b). If u and w are not in the same $\mathscr{X}$-residue, we use the non-trivial relator $r_u r_w^{-1}$, obtained after the elimination of u, to eliminate r_w, obtaining the same presentation with the same absence as above. Observe that in both subcases the $\mathscr{X}$-residues of u and w survive as a common $\mathscr{X}$-residue of H'. To conclude, note that the relator $b_{ij}^{\hat{w}} b_{ij}^{\hat{u}}$ in $B(H')$ is implied by $P_{\mathscr{X}}(H')$: replacing each x by r_x makes it disappear. □

7. COROLLARIES AND THE ROOTED GROUPS

Theorem 1 of Section 3 is clearly a particular case of the following corollary:

COROLLARY 1. *Let H and J be equivalent n-crystallizations. Then the groups $\xi_k^n(H)$ and $\xi_k^n(J)$ are isomorphic.*

Proof. By part (i) of the Strong Equivalence Theorem, we may suppose that J is obtainable from H by a finite sequence of crystallization moves, each of which involves color 0. Let us call the inverse of the fusion of a fusible edge y, the *antifusion creating* y. Note that each crystallization move restricted as above is either:

(i) the antifusion creating an edge y_1 followed by the fusion of a fusible y_2 with $\hat{\mathscr{Y}}_1 = \hat{\mathscr{Y}}_2 = \{0\}$:
(ii) the antifusion creating an edge y or the fusion of fusible edge y; in both cases with y having at least another edge with the same ends.

With this observation the result is straightforward from Theorem 2; the proof of Theorem 1 also finished. □

Corollary 2. *Let H be an n-crystallization. The isomorphism class of $\xi_k^n(H)$ does not depend on the ordering of the colors of H.*

Proof. It follows easily from the Switching Lemma that we can interchange two arbitrary colors i and j of H, by crystallization moves involving color 0. Just note that $(i, j) = (0, i) \circ (0, j) \circ (0, i)$ is a valid identity on permutations of Δ_n. With these arbitrary interchanges we may obtain any desirable permutation of the colors without changing the isomorphism class of $\xi_k^n(H)$. □

All the groups $\xi_k^n(H)$, $0 \leq k < n$, are peculiar in that the generators of each relator alternate ± 1 in their exponents. Call these presentations of abstract groups *alternating presentations*.

Lemma 5. *If $P = [G \mid R]$ is an alternating presentation for a group P and $a, b \in G$, then the quotient groups $[G \mid R \cup \{a\}]$ and $[G \mid R \cup \{b\}]$ are isomorphic.*

Proof. Let $\hat{G}$ be a set disjoint from G and in bijective correspondence with it via $g \to \hat{g}$. Note that

$$[G \mid R \cup \{a\}] \simeq [\hat{G} \cup G \mid R \cup \{a\} \cup \{\hat{x}bx^{-1} \mid x \in G\}].$$

Since R is alternating, R and $\hat{R}$, obtained from R by $\hat{x} \leftarrow x$, are equivalent. By using the relator $\hat{a}ba^{-1}$ to eliminate a we obtain

$$[\hat{G} \cup G^a \mid \hat{R} \cup \{\hat{a}b\} \cup \{\hat{x}bx^{-1} \mid x \in G^a\}],$$

where $G^a = G - \{a\}$. Now use the relator $\hat{a}b$ to eliminate b, and obtain

$$[\hat{G} \cup G^{ab} \mid \hat{R} \cup \{\hat{x}\hat{a}^{-1}x^{-1} \mid x \in G^{ab}\} \cup \{\hat{b}\}].$$

where $G^{ab} = G - \{a, b\}$. Note that each symbol in G^{ab} appears isolated in one relator. By using these relators we can eliminate all the symbols in G^{ab}, obtaining

$$[\hat{G} \mid \hat{R} \cup \{\hat{b}\}] \simeq [G \mid R \cup \{b\}]$$

and establishing the lemma. □

Let H be an $(n+1)^*$-graph. Define the *rooted groups* $\dot{\xi}_k^n(H)$, $0 \leq k < n$, by arbitrarily choosing a generator s of $\xi_k^n(H)$ and taking the quotient of this group by including the relator s in its presentation. The generator s is called the *root* of $\dot{\xi}_k^n(H)$.

Corollary 3. *The isomorphism class of $\dot{\xi}_k^n(H)$ does not depend on the choice of the root.*

PROOF. It is enough to note that the presentation for $\xi_k^n(H)$ is alternating, no matter whether H has one or two $\hat{0}$-residues. Then the corollary follows from Lemma 5. □

COROLLARY 4. *Let H and J be equivalent n-crystallizations. Then $\xi_k^n(H) \simeq \xi_k^n(J)$.*

PROOF. This result follows from Corollary 3 and from a repetition of the proof of Theorem 2 for $\xi_k^n(H)$. It is enough to choose as the root the vertex distinct from the ends of edge y at which we effect fusion in that proof. With this precaution the proof goes through and we obtain Theorem 2 for the rooted groups. Thus, in consequence, we also obtain Corollary 1, which for rooted groups is precisely the present corollary. □

THEOREM 3. *Let H be an n-crystallization. The group $\xi_{n-1}^n(H)$ is the fundamental group of $|K(H)|$.*

PROOF. From the definition of $\xi_{n-1}^n(H)$ we may use the relators in $P_k(H)$ to obtain only the $\{1, 2, \ldots, n-1\}$-residues as generators for this group. The only surviving relators are the root and the ones corresponding to $(0, n)$-gons. This presentation, following from the main result in [8], defines the fundamental group of $|K(H)|$.

Conceivably, all the groups presented are simply related to the fundamental group. There is empirical evidence that $\xi_k^n(H)$ is $n-k$ times the free product of $\pi_1(|K(H)|)$. This is the case in some small examples. A general proof has been elusive so far. A missing link seems to be an appropriate definitions of these groups for the class of all $(n+1)$-graphs.

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