JOURNAL OF COMBINATORIAL THEORY, Series B 30, 253–262 (1981)

A Minimax Theorem on Circuits in Projective Graphs

SÓSTENES LINS

Departamento de Matematica, Universidade Federal de Pernambuco, Recife–Pe. 50000, Brazil

Communicated by the Editors

Received October 4, 1978

A minimax theorem is proved. The theorem concerns packing non-separating circuits in eulerian graphs embedded in the projective plane. The proof includes a polynomial algorithm which produces a collection of edge-disjoint, non-separating circuits of the same cardinality as a transversal of such circuits.

1. INTRODUCTION

A *map* M is as a connected graph G, embedded in the topological sense, in a compact surface S, such that the following condition for well-embedded graph is met:

(WEG) Surface S minus graph G is a collection of disjoint open discs.

G is said to be then *graph of* M and is denoted by $\mathrm{gr}(M)$. S is called the *surface of* M and is denoted by $\mathrm{SURF}(M)$.

The disjoint open discs in $\mathrm{SURF}(M)\backslash\mathrm{gr}(M)$ are called the *faces* of M.

A *projective plane* is the compact surface obtained by identifying opposite points in the boundary of a closed disk.

A *projective map* is a map M such that $\mathrm{SURF}(M)$ is a projective plane.

A *corona* in a projective map M is a set c of edges of $\mathrm{gr}(M)$ which satisfies (a) and (b) below, and which is minimal relative to the verification of (b):

(a) $\mathrm{gr}(M)$, after the deletion of c, remains connected;

(b) condition (WEG) is not met after this deletion.

It can be shown that given a pair (M, c), where M is a projective map and c one of its coronas, there is a drawing of M in the disk (to be identified and to form a projective plane) such that only the edges in c appear in two pieces.

0095-8956/81/030253-10$02.00/0

When we do not identify the boundary, the associated planar map is called the *sun map induced by* (M, c), or simply, the *sun* (M, c).

Drawing *a* sun is the usual way of presenting a projective map.

Note that each edge e in c, corresponds, in the sun, to two edges incident to monovalent vertices. They are called the *semi-edges* of e.

A *cycle in a graph* is a subgraph which has even valency at all vertices. A *circuit in a graph* is a minimal cycle with at least one edge.

We do not distinguish explicitly between a cycle and its set of edges.

A *boundary* in M is a set of edges which form the boundary of some subset of faces of M.

It is well known that boundaries in M are cycles in $\mathrm{gr}(M)$.

An *nb-cycle* (non-bounding cycle) in a map M is a cycle in $\mathrm{gr}(M)$ which is not a boundary.

The *size* of a set or collection is its cardinality.

This paper proves the following theorem.

THEOREM 1.1. Minimax Theorem for eulerian projective maps. *In every eulerian projective map, the minimum size of a corona equals the maximum size of a collection of edge-disjoint nb-circuits.*

Before starting to prove this minimax equality we introduce the usual concept of dual map.

The *dual of a map M* is another map D obtained from M as follows:

(a) Place a dual vertex in the interior of each disk which is a component of $\mathrm{SURF}(M)\backslash\mathrm{gr}(M)$.

(b) For each edge e of $\mathrm{gr}(M)$ draw a corresponding dual edge which intersects $\mathrm{gr}(D)$ in just one interior point of e, and which links the vertices (which may coincide) in the interior of the discs (the two, again, may coincide) incident to e.

Observe that the dual of D is M itself. Also, there is a 1–1 correspondence between the faces of M and the vertices of D, and $\mathrm{SURF}(M) = \mathrm{SURF}(D)$.

The construction of the dual gives a natural 1–1 correspondence between the edges of M and the edges of D, and in this way they are identified.

The following proposition is useful.

PROPOSITION 1.2. Coronas and nb-circuits. *For every projective map M, coronas in M correspond to nb-circuits in D and vice versa.*

A proof of this statement is not given in this paper. From the topological definitions of a map it is intuitively clear. A formal proof for a combinatorially defined map can be found in [3].

Using Proposition 1.2 we obtain a theorem dual to Theorem 1.1 by formally interchanging the terms corona with nb-circuit and vertex with face.

THEOREM 1.3. Dual projective minimax theorem. *For every projective map which has all its faces of even valency, the minimum size of an nb-circuit equals the maximum size of a collection of edge-disjoint coronas.*

A more useful consequence of Proposition 1.2, relative to the proof which we give in Section 2, is the following form, obtained by partial dualization of Theorem 1.1.

THEOREM 1.4. Primal-dual minimax theorem. *For every eulerian projective map M, the minimum size of an nb-circuit in D equals the maximum size of a collection of edge-disjoint nb-circuits in M.*

A proof of the Theorem in the above form is given in the next section.

2. PROOF OF THEOREM 1.4

PROPOSITION 2.1. Intersection of nb-circuits in M and D. *If M is a projective map, then every nb-circuit in D has non-null intersection with every nb-circuit in M.*

The proof of this statement is also not supplied here. Topologically it is clear. A formal and more general proof of the statement, using a completely combinatorial terminology, can be found in [3]. In fact, there it is proved that the intersection has odd cardinality.

As a consequence of 2.1 we have the following lemma.

LEMMA 2.2. Lower bound lemma. *For projective map M, the minimum size of an nb-circuit in D is at least equal to the maximum size of a collection of edge-disjoint nb-circuits in M.*

We now introduce some notions used to establish that equality holds in Theorem 1.4.

A map induces a cyclic ordering of the edges incident to the vertices of its graph. If a vertex, v, is even valent, having valency $2n$, say, then the notion of *opposite edges at v* is well defined. The adjective refers to a pair of edges that differ by n in a sucessive numbering (of the edges incident to v) that follow the cyclic ordering induced by the map.

A *smooth cycle*, or simply *sm-cycle*, in an eulerian map M is a closed path in $\mathrm{gr}(M)$ with the property that every two of its consecutive edges are opposite at the common vertex.

Every edge of $\mathrm{gr}(M)$ is in exactly one sm-cycle.

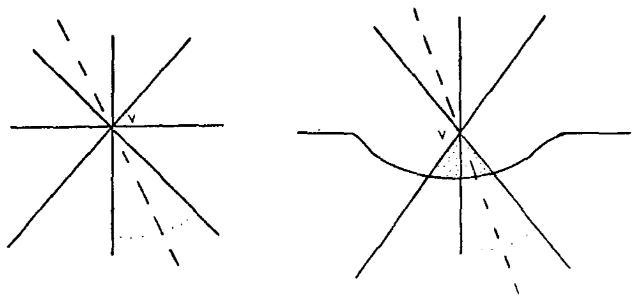

FIGURE 1

We now define the concept of *topological perturbation*. Given an even valent vertex v, of valency at least 6 in a map M, and a pair of opposite edges at v, the topological perturbation of the pair at v is the local deformation of them in order to miss v, as shown in Fig. 1.

Note that after a perturbation at v, the valency of v is decreased by two. Also, new 4-valent vertices, as well as new triangular faces, are created. However, we have the following:

LEMMA 2.3. Lemma on topological perturbation. *Topological perturbation does not change the size of a minimum nb-circuit in the dual map.*

Proof. Let the dual pair of maps before perturbation be denoted by M and D, and the perturbed pair of dual maps be denoted by N and E.

Take a minimum nb-circuit in E and call it c. If c does not use vertices corresponding to the new triangular faces formed by the perturbation, then c is also an nb-circuit of D and there is nothing to be proved.

If c uses such vertices, then one of the two suitable revisions illustrated in Fig. 2 gives a cycle of E, denoted by d, which uses fewer of the new vertices and is of the same size as c. Since d is obtained by the symmetric difference of c and a bounding circuit in gr(E), it follows that d is an nb-cycle. Also, since c is minimum and d is of the same size as c, it follows that d is a circuit. It is easy to deduce that at most two such revisions produce an nb-circuit in E which is of the same size as c, which is also an nb-circuit in D. This concludes the lemma. ∎

Note that every nb-circuit in N remains so in M if the new triangular faces created by the perturbation are contracted to a point. As a consequence of this observation and the above lemma, Theorem 1.4 is true for M if it is true for N.

We are going to use topological perturbation in the splitting of digon configurations that we now define. Assume that two segments, p and q, of sm-cycles (which may be the same) cross each other twice at vertices v and

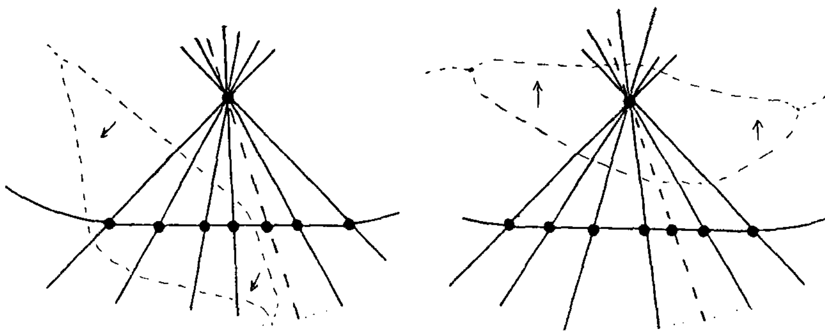

FIGURE 2

w forming a bounding digon, which is usually subdivided by bivalent vertices.

In the above situation p and q induce a *digon configuration*, $DC = DC(p, q)$, defined as the planar submap formed by

(a) the edges of p and q between the crossings v and w, which form the boundary of the digon;

(b) the segments of sm-cycles that are in the interior of the digon;

(c) all the edges exterior to the digon which are incident to its boundary.

The crossing points v and w of p and q are called the *corners* of the digon configuration. The edges in (c) are called the *outer* edges of the digon configuration.

Our intention is to free ourselves from digon configurations. Our next definition is a preparation for this.

The *splitting* of a 4-valent vertex v in a map M is the separation of the four edges incident to v into two subsets of two edges each, provided opposite edges are not incident after the separation.

Hence, there are two distinct ways to effect the splitting and they both can be done in SURF(M). After a splitting at v is effected, two bivalent vertices replace v.

The *splitting of a digon configuration*, $DC(p, q)$, is defined by the following operations:

(a) if at least one of the corners of DC(p, q) is 4-valent, call it v and proceed to (c);

(b) if (a) is not true, choose one of the two corners of DC(p, q), calling it v. Apply topological perturbation (in either direction) at the pair of opposite edges at v, one of which is the edge of p (or q) incident to v. This topological perturbation clearly induces a digon configuration for which (a) is true;

(c) effect the splitting at v which does not disconnect the subgraph induced by the edges of the digon configuration.

A digon configuration DC is *minimal* if the set of edges of any other digon configuration is not included in the set of edges of DC.

LEMMA 2.4. Lemma on digon configurations. *Assume that M is a projective map and $DC(p, q)$ is a minimal digon configuration in M. Then the splitting of DC does not change the minimum size of an nb-circuit in D.*

Proof. Let N be the projective map obtained from M after the splitting of DC and E, its dual.

From Lemma 2.3 it follows that we may suppose condition (a) to be true. In this case the modification implied by the splitting of DC is that two faces of M, say f and g (the latter inside DC), become a unique face f' in N.

Take a minimum nb-circuit c in E. We may assume that c is incident to f'; otherwise c would be an nb-circuit of D and there is nothing to be proved.

Consider the path s defined as the part of c which starts of f' and proceeds in the interior of the (former) digon up to a vertex h, which is the first face not in the interior of the digon. Also consider the path t which goes from f' to h along p or q, but in the exterior of the digon, as shown in Fig. 3.

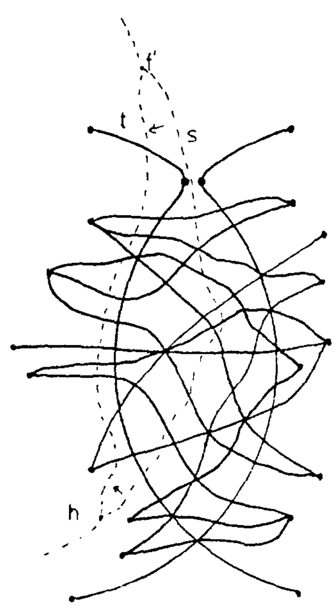

FIGURE 3

The paths s and t between f' and h constitute a circuit in E which is clearly a bounding circuit. This implies that c', defined as the symmetric difference of c and this circuit, is a nb-cycle in E, which is also in D.

What remains to be proved is that the number of edges in c' is the same as the number of edges in c. For this purpose we identify and edge as the crossing of the edge with its dual edge.

The revised path t crosses one edge incident to the vertex that is being split. This crossing is compensated for by the crossing of the old path s with the boundary of the digon in DC. All the other crossings of the revised path are compensated for as follows. The minimality of DC and Jordan's theorem for simple curves on the plane imply that for each crossing of t with an outer edge of a segment that crosses the digon, we have an odd number of crossings of s with the same segment in the interior of the digon. In fact, since we are assuming c of minimum size, this odd number is always one.

Therefore c' and c have the same number of crossings. This shows that c' is an nb-circuit in D of the same size as c, and proves the lemma. ∎

After the splitting of a vertex is effected, the circuits in the resulting map, N, are also cricuits of the original map, M. Therefore, if N is obtained by splitting a minimal digon configuration in M, Theorem 1.4 holds for M if it holds for N.

By repeating the splitting of minimal digon configurations we may suppose that we have a map without digon configurations at all.

To finish the proof of Theorem 1.4 for maps without digon configurations, we first need a definition.

An *sm-path in a sun* (M, c) is either an sm-cycle in M, which does not contain edges in c or a segment of an sm-cycle of M which starts at a monovalent vertex incident to one-semi-edge and finishes in another.

As our map (call it M again) has no digon configurations, the sm-paths of (M, c) for every corona c are n paths between monovalent vertices, where n is the number of edges in c.

Take any corona c, with n edges say, and draw the sun (M, c). Two possibilities can occur:

(a) every two sm-paths of (M, c) cross once;

(b) there exist two sm-paths of (M, c), p and q, that do not cross.

In case (a) the proof of Theorem 1.4 is complete because the n sm-paths link opposite points and correspond to a collection of edge-disjoint nb-circuits in M and c is an nb-circuit in D of size n.

In case (b) we can easily produce an nb-circuit d in D, which does not cross p or q and crosses every other sm-path at most once. The justification for the existence of d follows by straightforward induction on the number of sm-paths; it uses the fact that M is free of digon configurations. See Fig. 4a,

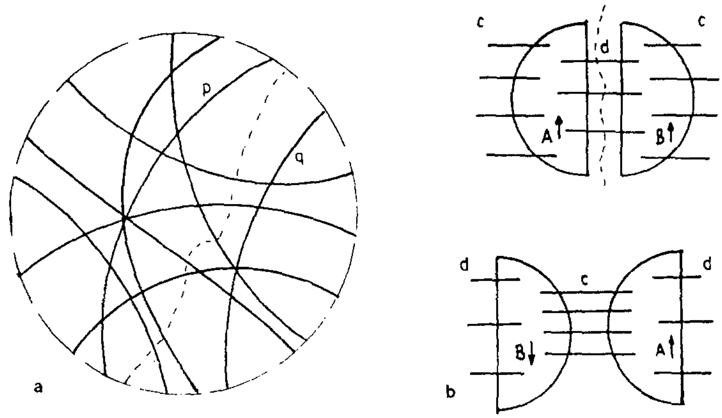

FIGURE 4

where d is presented in broken lines. The effective determination of this circuit is of fundamental algorithmic importance. It can be obtained with simplicity only because, again, M is free of digon configurations.

The nb-circuit d corresponds to a corona in M which has size at most $n-2$. Find a drawing for the sun (M, d) and check for (a) and (b) again. See Fig. 4b. Iteration of this procedure must end when, for some sun (M, c), condition (a) is satisfied. This concludes the proof of Theorem 1.4 and of the equivalent statements Theorems 1.1 and 1.3. ∎

3. CONCLUSION

When we apply the inverses of the splittings and topological perturbations along with the revisions of the nb-circuit in the dual, the method of proof of Theorem 1.4 provides an algorithm to present the objects involved in the minimax equalities. All the identifications of structures and operations are made in polynomial time relative to the size of the map, hence we have a good algorithm. We name it the *splitting algorithm.*

The splitting algorithm can be used in the context of multicommodity flow problems. For example, it provides an algorithmic proof of the main theorem in [4], as pointed out to us by P. Seymour.

As a corollary to Theorem 1.4, we prove results similar to Theorems 1.1 and 1.3 where the requirements of being eulerian and having even valency faces are dropped.

A collection of subsets, repetitions allowed, is said to form an *n-matching* if the total number of occurrences of every object in the collection is at most n.

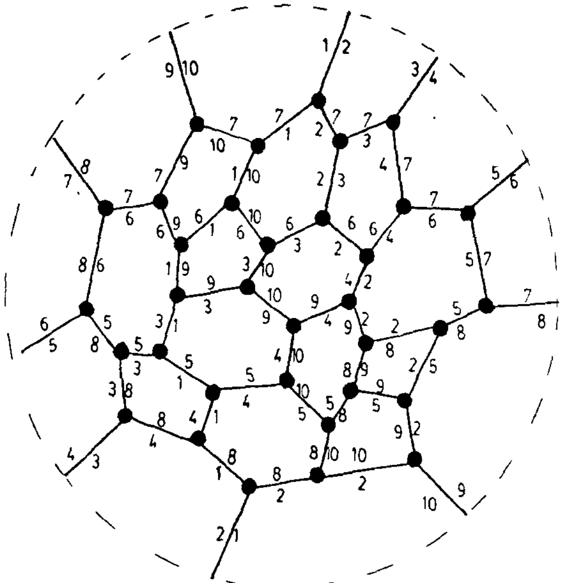

FIGURE 5

THEOREM 3.1. Minimax theorem for general projective maps. *For every projective map, the minimum size of a corona equals half the size of a maximum 2-matching of nb-circuits.*

THEOREM 3.2. Dual minimax theorem for general projective maps. *For every projective map, the minimum size of an nb-circuit equals half the size of a maximum 2-matching of coronas.*

Proofs. Theorem 3.2 is equivalent to Theorem 3.1 by formal dualization. To prove Theorem 3.1, double each edge forming a bounding digon in each case, thus obtaining an eulerian map. Observe that all the circuits of the new dual are doubled in size but remain in 1–1 correspondence with the circuits of the original dual. Application of Theorem 1.4 to the resulting map trivially implies, Theorem 3.1. ∎

We conclude by presenting an example of the application of the splitting algorithm: Figure 5 shows a projective map with five as the minimum size of a corona, fact which is proved by the 10 nb-circuits with the property of 2-matching.

ACKNOWLEDGMENTS

We are indebted to the University Federal of Pernambuco and to CAPES, both of Brazil, for financial support. We would like to thank P. Seymour of Oxford University for finding an error in an extension of the result here presented. Our thanks are also extended to D. H. Younger, of the University of Waterloo, and to the referee for many valuable suggestions concerning the presentation.

REFERENCES

1. J. EDMONDS, A combinatorial representation for polyhedral surfaces, *Notices Amer. Math. Soc.* **7** (1960), 646.
2. W. T. TUTTE, What is a map? *in* "New Directions in the Theory of Graphs" (F. Harary, Ed.), Academic Press, New York, 1971.
3. S. LINS, "Graphs of Maps," Ph.D. thesis, University of Waterloo, 1980.
4. P. D. SEYMOUR AND H. OKAMURA, Multi-commodity flows in planar graphs, *J. Combin. Theory Ser. B*, **31** (1981), 75–81.
5. L. LOVÁSZ, 2-matchings and 2-coverings in hypergraphs, *Acta Math. Acad. Sci. Hungar.* **26** (1975), 433–444.