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A SEQUENCE REPRESENTATION FOR MAPS

Sostenes LINS

Departamento de Matematica, Universidade Federal de Pernambuco, Recife—Pe. 50000, Brazil

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An easy implementable polynomial algorithm to test for isomorphism of graphs embedded in arbitrary compact surfaces (maps) is given.

The maps are defined algebraically by Tutte's axiom system. We produce a canonical codification of them as a sequence of $4e$ integers ($2e$ if the map is orientable) where e is the number of edges. To test for isomorphism between two maps we just have to compare their codes.

Some applications relying on the implementation are given.

1. Introduction

A map (of a graph) is usually presented as a connected graph G embedded in a compact surface S such that the topological space $S\backslash G$ is a collection of disjoint open discs.

In [1] Tutte introduced the algebraic counterpart that we describe next.

A *map* is an ordered set of 3 permutations, (M, T, L), acting on a finite non-empty set B, which elements are called *vanes*, satisfying the following conditions:

(or1) $TL = LT$.

(or2) $T^2 = L^2 =$ Identical permutation.

(or3) For each x in B, the elements x, Tx, Lx and TLx are distinct.

The way that M is related to the system is through the axioms:

(mp1) $MT = TM^{-1}$.

(mp2) $M^k x \neq Tx$, for every integer k and x in B.

If the following axiom holds, then we have a connected map.

(mp3) $\langle M, T, L\rangle$ is transitive on B, where $\langle M, T, L\rangle$ denotes the group of permutations generated by M, T and L.

If $\langle M, T, L\rangle$ acting on B is a map, then we say that M is a (T, L)-map. If no confusion arises, the reference to T, L and B are suppressed and we say simply map M. Because of this simplification, M is called the *main permutation* of the map. T and L are called, respectively, *transversal and longitudinal orientations* of M.

As a simplification in the notation we write x' for Tx and $-x$ for Lx.

Axioms (or1) and (or3) imply that the orbits of the group of permutations generated by T and L, $\langle T, L\rangle$, have 4 elements each. Each such orbit is called an *edge*. In particular this implies that the cardinality of B is divisible by 4. The set of all edges of M is denoted by $E(M)$. Edge $\{x, x', -x, -x'\}$ is denoted by $/x/$.

Given x in B, the cycle of x under M is denoted by orb (x, M). From (mp2) it follows that orb (x, M) and orb (x', M) are disjoint, for any x. The unordered pair $\{\text{orb}(x, M), \text{orb}(x', M)\}$ is called a *vertex* of M. Since one of the orbits completely determines, by (mp1), the other, which we call its *opposite*, we speak of the vertex orb (x, M) even if logically not precise. The set of vertices of M is denoted by $V(M)$.

The *graph* of a g-map M, gr (M), is the graph where the vertices and edges are those of M, such that the ends of edge $/e/$ are orb (e, M) and orb $(-e, M)$.

A connected map M is called *orientable* if $\langle M, TL\rangle$ has two orbits on B. The alternative is to have one orbit and in this case M is said to be non-orientable. The *orientability label*, OL, is defined as $\text{OL}(M)=1$, if M is orientable and $\text{OL}(M)=0$, if it is not.

If M is a (T, L) map, then $D=MTL$ is an (L, T)-map called the dual of M. See [1].

The *Euler characteristic*, EC, of a map M is the integer

$$\text{EC}(M)=|V(M)|+|V(D)|-|E(M)|.$$

The pair $(\text{OL}(M), \text{EC}(M))$ is called the *combinatorial surface* of M and denoted by SURF (M).

We now sketch briefly the connection between the algebraic and topological maps.

A connected graph G embedded in a compact surface S defines a map M with gr $(M)=G$ and SURF $(M)=S$, the converse being also true.

We give only an example of this relation. For more detailed treatment we refer to Tutte's original paper [1] or to [2]. Consider the map given by the following scheme, where each row represents an orbit of a vertex.

vertex u	1	-7	6	$-11'$	$10'$
vertex v	-1	$-8'$	$-9'$	2	$-15'$
vertex w	-2	$-14'$	3	12	7
vertex x	-3	$13'$	$8'$	$-10'$	$4'$
vertex y	-4	15	$14'$	$5'$	$11'$
vertex z	-5	$9'$	-6	$-13'$	$-12'$

It can be seen, by computing the dual, that $\text{EC}(M)=-3$ and also, since $\langle M, TL\rangle$ is transitive on B, that $\text{OL}(M)=0$.

We present an embedding of gr (M) in the non-orientable surface of Euler characteristic -3. See Fig. 1.

Note that each edge has a double orientation, the 4 combinations of them forming its 4 vanes. The longitudinal orientation is reversed by the application of

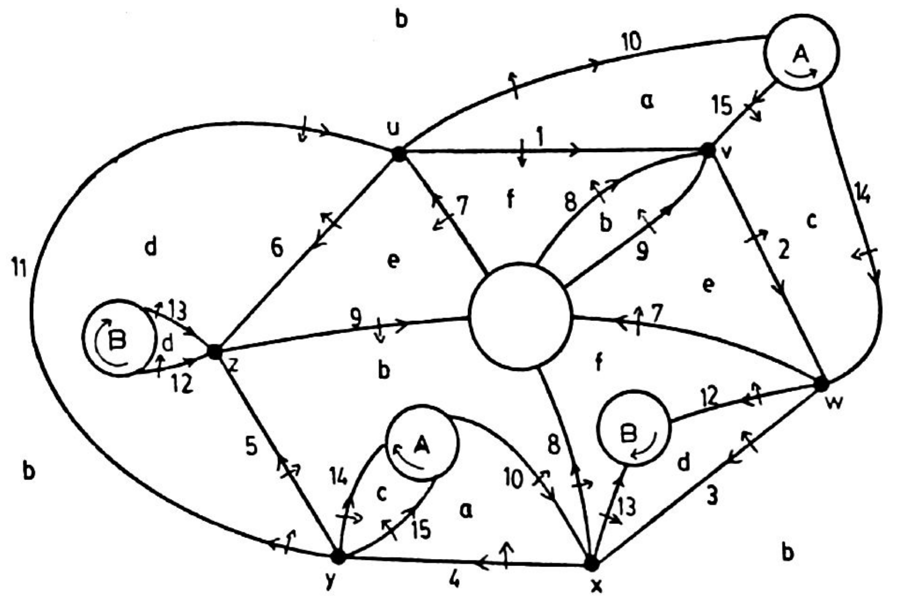

Fig. 1.

L and so is the transversal orientation by the application of T. M acts by rotating a vane to the next by using the tail of the edge as axis of rotation, rotating in the direction indicated by the crossing arrow. The hole without label is a cross-cap, the ones labeled A form an orientable handle and the ones labeled B form a non-orientable handle.

A simple inductive approach provides a general algorithm to go from the scheme of a combinatorial map to its drawing in the correspondent topological surface. For more detailed analysis we refer to [2].

2. Computing the symbol

All the maps of this section are connected.

Given two maps, (M, T, L) acting on B and (N, X, Y) acting on C, we say that they are isomorphic if there exists a bijection i between B and C such that for all vanes x in B we have:

(is1) $i(Mx) = N(i(x))$.

(is2) $i(Tx) = X(i(x))$.

(is3) $i(Lx) = Y(i(x))$.

This definition agrees with the intuitive topological idea, which is formalized in terms of homeomorphism between the two surfaces such that the image of one graph is the other. The combinatorial definition is, however, much more easy to work with since it provides an immediate answer for the question of deciding if two maps are the same, as we show.

A *rooted map* is a pair (M, b) where M is a map and b is one of its vanes. This rooting is equivalent to the one used by Tutte in his enumerative work: That is because we can associate with a vane an edge (its supporting edge), a vertex (its tail) and a face (the face which contains the tail of its crossing arrow).

Next proposition shows the importance of rootings.

2.1. Proposition (Rooting and isomorphism). *An isomorphism between two maps is determined by the knowledge of the image of one vane.*

Proof. As we are dealing with connected maps, given any two vanes x and y, there exists a *word* f, formed by M, T and L (meaning composition of permutations) such that $f(x)=y$.

Assume that we are under the hypothesis of the proposition, that is, we have an isomorphism i between map (M, T, L) acting on B and map (N, X, Y) acting on C, and we know that $i(b)=c$.

Given vane x in B, find f, word in M, T and L, such that $f(b)=x$. Word f corresponds to a word g in N, X and Y, formed by replacing M for N, T for X and L for Y. The definition of isomorphism for maps implies then

$$i(x)=g(i(b))=g(c).$$

This proves the proposition. □

We say that rooted map (M, b) is isomorphic to rooted map (N, c) if the maps M and N are isomorphic and the image of b is c under the isomorphism.

As previous proposition shows, unlike for graphs, the isomorphism problem for rooted maps is simple in principle. However it is not clear how to get the various words f to get the image of the other vanes in two isomorphic rooted maps. There are many alternatives to get those. We present an indirect systematic method which is easy to implement in a computer and that is also conceptually simple.

Given a rooted map (M, r) we start by attaching numbers to exactly half of the vanes of M by means of the algorithm that follows. It uses the following concepts:

A stack s.

The notion of a current vane z.

The current number n.

2.2. Rooted Numbering Algorithm. (a) Give to the root r the number 1. Make $n=1$ and stack s empty. Go to step (b).

(b) If a vane x just received number $2n-1$, then
  (i) Put x on the top of s.
  (ii) Give the number $2n$ to vane $-x'$.
  (iii) Make $-x'$ the current vane z.
  (iv) Go to step (c).

(c) To find the vane with number $2n+1$ look for the first M-successor of z, say

x, such that x and x' are not yet numbered. We have two cases:

(c1) If such M-successor is found:

(i) give to x number $2n+1$.

(ii) increase n by 1.

(iii) go to step (b).

(c2) If there is no such M-sucessor:

(i) If s is not empty make z equal to the top of s and remove this element from s. Try step (c) again.

(ii) If s is empty, then z must be b and the rooted numbering algorithm is complete. □

Clearly the above algorithm is polynomial: Every vane enters the stack s at most 1. (Precisely 1 for the odd numbered and 0 for the even numbered.) The number of times that we apply M and TL before some modification of the stack is accomplished is bounded by the maximum valency of a vertex.

The following important proposition is clear from the definitions.

2.3. Proposition (Rooted numbering and isomorphism). *Bijection i is an isomorphism between (M, b) and (N, c) if and only if for any vane x of M and the respective rooted numberings* $\text{numb}(x) = \text{numb}(i(x))$.

Proof. Immediate. □

For two examples of the rooted numbering algorithm we refer to Fig. 2. There is presented two maps, one planar and the other in the projective plane. (Or PP for short.)

The vanes numbered are the ones for which we present the transversal

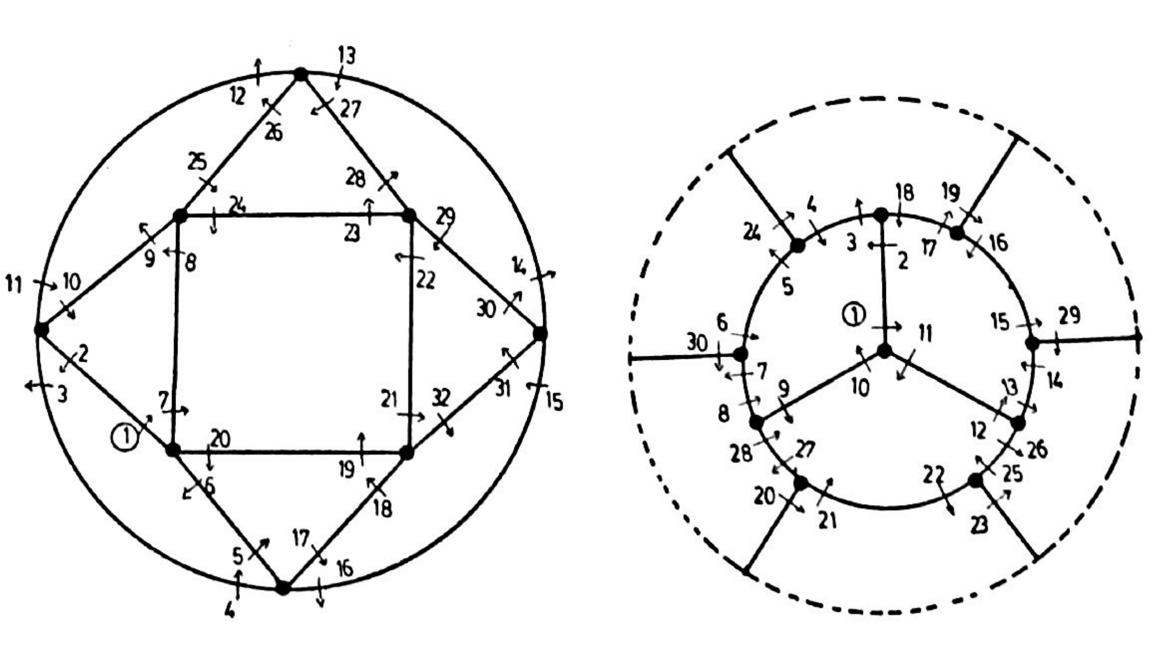

Fig. 2.

orientations. Their longitudinal orientations are always outdirected and so are not presented. Observe that as a consequence of the rooted numbering algorithm, vane x is numbered if and only if vane x' is not numbered. Also the numbers attached are $1, \ldots, 2e$ where e is the number of edges.

Note that for the planar one all the transversal orientations are clockwise, while this is not the case for the projective. Next proposition will deal with this observation.

2.4. Proposition (Rooted numbering in orientable maps). *A map M is orientable if and only if all the vanes which receive number by the rooted numbering algorithm (with an arbitrary root) are clockwise (or counterclockwise) in a drawing for M.*

Proof. The algorithm numbers the root and proceeds numbering vanes obtained from the root by iterated application of TL and M. Therefore the vanes which receive numbers form an orbit of $\langle M, TL \rangle$ if and only if M is orientable. The interpretation of the crossing arrows in the drawing, then, proves the proposition. □

We want to use the rooted numbering of a rooted map to form a canonical symbol for the latter. From the symbol we are able to recover the rooted map with the vanes labeled in a canonical way. A summary of the process is the following:

Scheme with arbitrary labels

①

Rooted numbering

②

Canonical symbol

③

Scheme with canonical labels

Up to now we have only explained the first step. Before going into the second we want to define what we mean by canonical labeling for the vanes.

The set of labels is the subset of integers from 1 to $4e$.

The canonical labeling, by definition, satisfies the following:

$$Tx = x+1 \quad \text{if } x \text{ is odd,}$$
$$= x-1 \quad \text{if } x \text{ is even.}$$
$$Lx = x+2 \quad \text{if } x = 1 \text{ or } 2 \bmod 4,$$
$$= x-2 \quad \text{if } x = 0 \text{ or } 3 \bmod 4.$$

The above requirements do not depend on the numbers of the vanes given by the rooted numbering algorithm. However we also require that vane with number $2x-1$ must be labeled $4x-3$.

Now it is an easy matter to check that the rooted numbering uniquely defines labels, for all the vanes. These labels, attached in the above way, are called *canonical labeling.*

The canonical symbol is obtained as follows from the rooted numbering. We use two functions $p, q: 2E \to 2E$, where $2E$ is the subset of integers from 1 to $2e$. For all vanes x which receive a number we define:

$p(\text{numb}\,(x)) = \text{numb}\,(Mx)$ or $\text{numb}\,((Mx)')$ whichever exists. Also for every vane x which did not receive a number we define:

$q(\text{numb}\,(x')) = \text{numb}\,(Mx')$ or $\text{numb}\,((Mx')')$ whichever exists.

The *canonical symbol* for the rooted map is then, by definition, the finite sequence with $4e$ elements:

$$(p(1), p(2), \ldots, p(2e); q(1), q(2), \ldots, q(2e)).$$

We observe that in a drawing $p(x)$ is obtained by looking at the number of the dart pointed by the head of the crossing arrow of dart numbered x. Analogously $q(x)$, replacing head by tail. This observation together with Proposition 2.4 imply that p and q are bijections if and only if M is orientable. If that is the case, then p and q are inverses permutations and the canonical symbol can be simplified to $(p(1), p(2), \ldots, p(2e))$.

The above concludes our considerations about step 2. Now comes the more important, yet simple, step 3.

2.5. Proposition (Recoverability). *A rooted map is recoverable from its canonical symbol.*

Proof. The map will be recovered with the canonical labeling for the vanes. Since this accounts for B, T and L we just have to recover M.

The equality $p(x) = y$ implies exactly one of the following:

(a) $M(2x-1) = 2y-1$, if x and y are both odd.

(b) $M(2x-1) = 2y$, if x is odd and y is even.

(c) $M(2x) = 2y-1$, if x is even and y is odd.

(d) $M(2x) = 2y$, if x and y are both even.

Analogously the equality $q(x) = y$ implies exactly one of the following:

(a) $M(2x) = 2y$, if x and y are both odd.

(b) $M(2x) = 2y-1$, if x is odd and y is even.

(c) $M(2x-1) = 2y$, if x is even and y is odd.

(d) $M(2x-1) = 2y-1$, if both x and y are even.

The two implications above will account for the value of M in its $4e$ vanes proving the proposition. □

We have defined the symbol for a rooted map. However, since it is a finite sequence of integers we can easily define the *symbol for the map* itself. Take it as, for instance, the lexicographical maximum among the symbols for its rooted maps.

3. Applications

We have implemented the algorithm to compute the symbol of a map and we would like to briefly outline a number of possibly interesting applications.

3.1. *Isomorphism of graphs*

Several papers have been written about isomorphism of planar graphs, as for instance, [4] and [5]. The technique that gives polynomial algorithms consists in considering planar drawings of the graphs and test for isomorphism of maps. The point is that by Whitney's Theorem, [3], if the graph is vertex 3-connected the map is unique. Extensions of this method permitting to deal with all planar graphs appear in [5].

The same techniques extend beyond the plane if we are able to produce classes of graphs for which the number of embeddings in a given surface is small and produced by a polynomial algorithm. For the classes of graphs such that this assumption is true, we can decide isomorphism in polynomial time by comparing the symbol of one embedding of the first graph with the symbols of all the embeddings of the second.

We do not want to leave the topic without giving a specific example of a class of graphs for which the above assumption is true.

A connected graph is said to be *cyclically n-connected* if the smallest (with respect to number of edges) coboundary such that after its deletion both components contain cycles has n edges.

Consider the class CP^4 of the cubic cyclically 4-connected graphs which are graphs of projective maps and contain a subgraph homeomorphic to the Petersen graph. The Petersen graph with labels on the vertices embeds in exactly two distinct ways in the projective plane. See Fig. 3.

To embed in PP a graph in the class CP^4 we proceed as follows. As we have a subgraph homeomorphic to the Petersen graph we can proceed from it to get the given graph by subdividing the faces of an embedding of it, being careful not to create any (subdivided by bivalent vertices) mono, bi, or triangular faces. This is possible by the assumption that the graph is cyclically 4-connected.

The first such subdivision decides which of the two is the embedding to proceed: that follows by the perfect symmetry (only one rooted map) of the embeddings of the Petersen graph in PP. Essentially the unique subdivision is shown in the embedding on the left. By the symmetry it can be supposed between edges 1, 9 and 2, 10 as it is shown in Fig. 3. Note that this subdivision is not possible in the embedding on the right. That means that any path that we put on

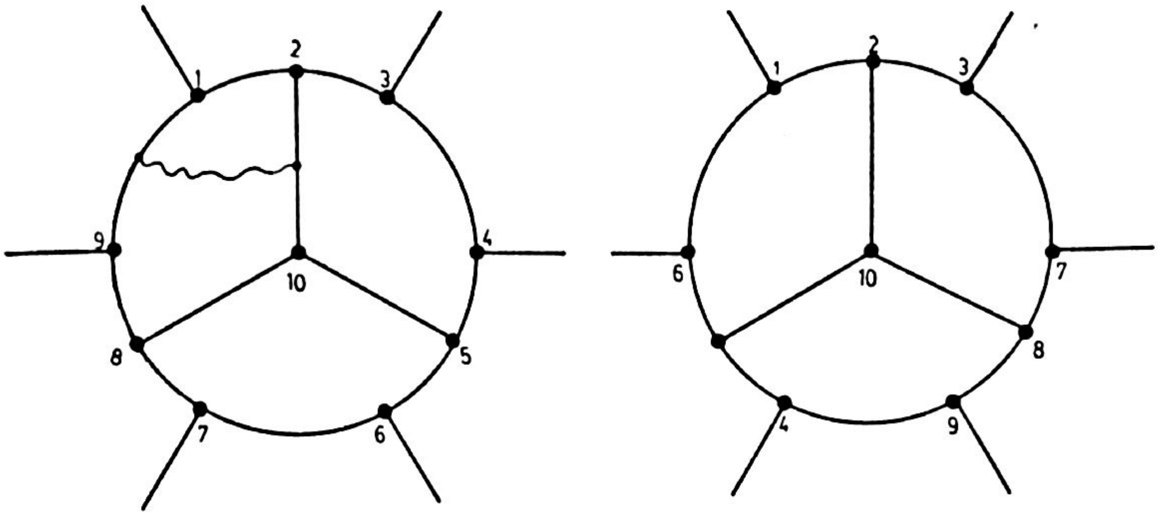

Fig. 3.

the graph will have to go through a unique face of the map and from this point on everything is forced. Follows that the isomorphism test for graphs in CP^4 is polynomial. A recursive construction for the graphs in CP^4 is given in [9].

Recently a polynomial algorithm to test for embeddability of graphs in the projective plane (possibly extendable to the torus) was devised by Younger [6]. It is conceivable that this algorithm together with the techniques developed in [5] and the idea of symbol for a map might be put together to solve polynomially the isomorphism problem for general projective graphs or even graphs in surfaces of higher connectivity. The matter is not carried any further in this paper.

3.2. *Construction of catalogs*

The symbol generated by the algorithm explained is suitable for the generation of catalogs of specific classes of maps, as did in [7]. In practice the search for the symbol of the map among the $4e$ possibilities can be considerably reduced by means of "ad hoc" conventions such as to have the distinguished root incident to a vertex of maximum valency or to the face of smallest valency, etc. The sequence representation permits convenient storage and fast identification of duplicates by means of binary search trees or their variants.

3.3. "*Enantiomorphic*" *forms in the Heawood map*

The Heawood embedding of seven countries in the torus has two distinct rootings. This fact is interesting because it is, at least to the author, very non-intuitive. Apparently we have perfect symmetry. Fig. 4 shows the two different numberings.

The symbols agree up to the 20th coordinate:

$$(7, 3, 11, 5, 15, 1, 19, 9, 23, 2, 27, 13, 31, 4, 35, 17, 39, 6, 42, 21, 30, \dots)$$
$$(7, 3, 11, 5, 15, 1, 19, 9, 23, 2, 27, 13, 31, 4, 35, 17, 39, 6, 42, 21, 36, \dots)$$

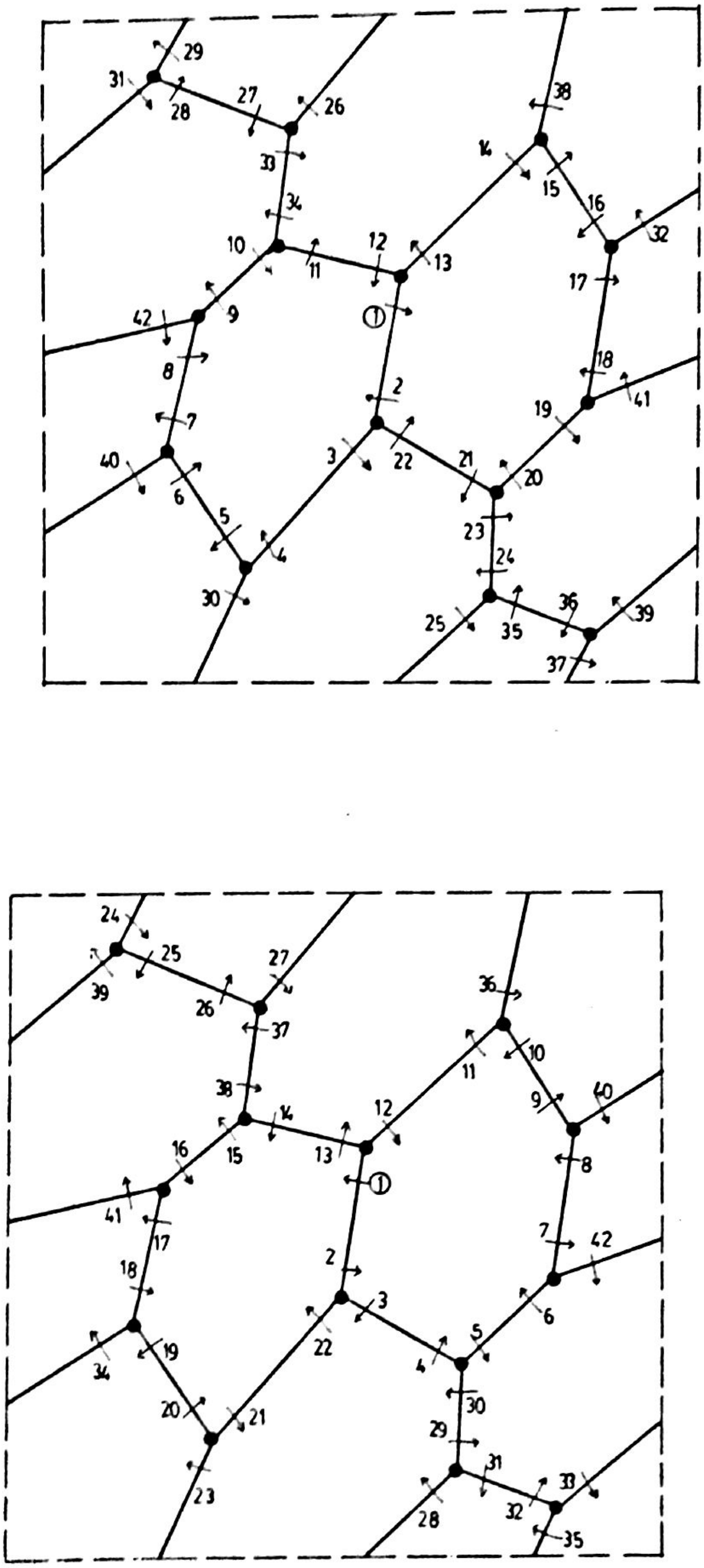

Fig. 4.

3.4. *Finding automorphism groups and symmetrical drawings*

The symbol for rooted maps can be used for polynomial computation of the automorphism group of the map. Just look for the roots that give lexicographical maximum symbol. Those are precisely the equivalent vanes and they induce the automorphism group of the map. We can describe this group as group of permutations of vanes, edges, vertices, faces, etc, whichever is more convenient.

The smallest set of elements which permits a nondegenerated description is usually the better. For instance the automorphism group of the projective map of Fig. 5 as face permutations is:

1	2	3	4	5	6	7
2	1	3	6	5	4	7
1	2	7	6	5	4	3
2	1	7	4	5	6	3

It is non-degenerated because all the 4 permutations are distinct, and we can see that exactly 4 vanes give maximum symbol: $1, 2, 3, 4$.

The drawing on the top of Fig. 5 has no apparent symmetry. However using the information about the symmetry on the faces we are able to produce the drawing on the right which shows all the symmetries in the planar drawing.

This fact is easily generalized: given the automorphism group of a map it is possible to find a planar drawing showing at least a non-trivial subgroup of its automorphism group. The technique was extensively used in [7] for the construction of the drawings of first part of the planar maps generated.

3.5. *Two distinct triangular embeddings of* K_{12}

Sometimes in the solution of the Heawood map conjecture, by using distinct groups, two triangular embeddings of the same complete graph in (necessarily) the same orientable surface are produced.

A specific case of two triangular embeddings of K_{12} is presented in [8, pp. 344–345]. The duals of these maps consists of 12 faces which are 11-gons mutually adjacent by exactly one edge forming a cubic graph with 44 vertices. Are the maps different? Or is it possible to find an isomorphism between them? To answer this question we have translated the embeddings given in the above reference into our terminology and computed the symbols for the maps.

The choice of K_{12} was motivated by the fact that 12 is the next regular orientable case after 7. We believe that K_7 (dual of Heawood map) is known to be uniquely embeddable in the torus.

Let us call M and N the two embeddings of [8]. Just as a matter of reference, we give the vertices, the faces and the symbols (lexicographical maximum) for M and N. The information is, of course, doubly redundant. However the redundancy might be used to check the algorithm presented in this paper.

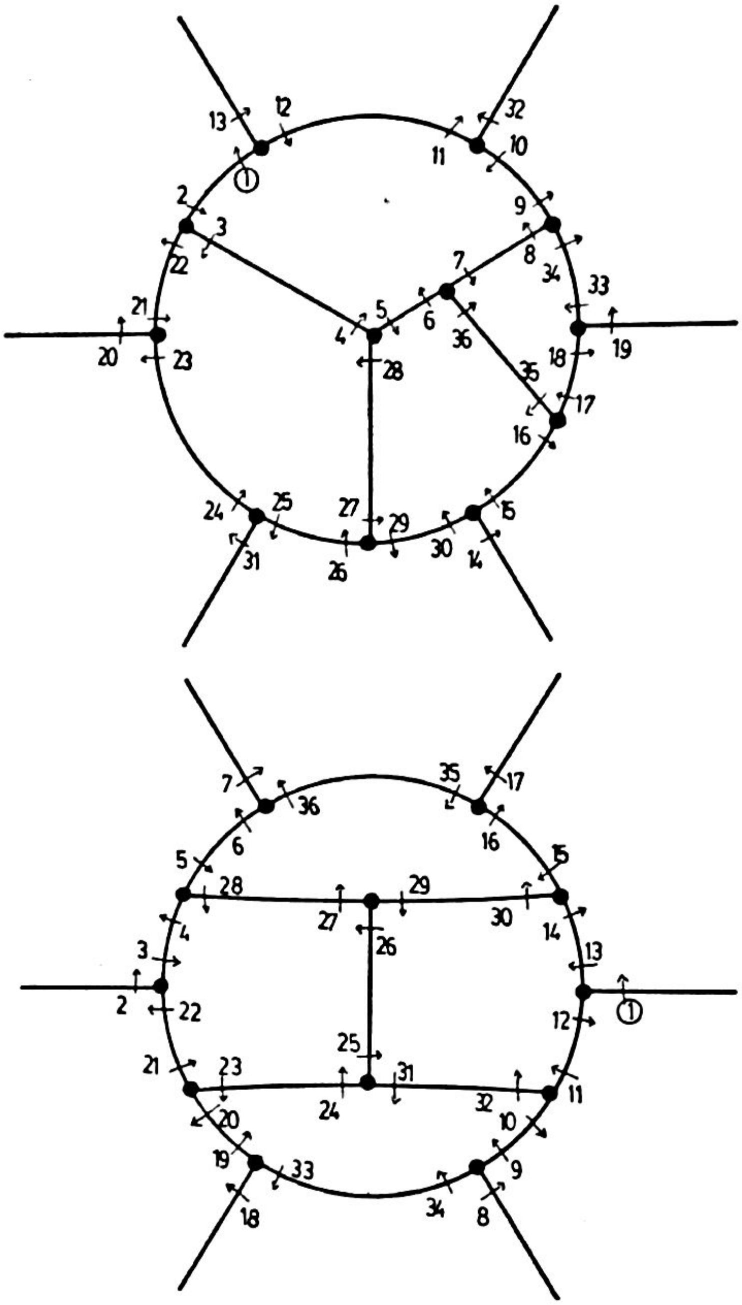

Fig. 5.

Vertices of first embedding: *M*

V.1	−3	−9	−21	−32	−48	49′	33′	22′	10′	4′	1′
V.2	−1	−5	−13	28′	−27	36′	−35	−54	14′	6′	2′
V.3	−2	−7	−17	−24	−38	−65	39′	25′	18′	8′	3′
V.4	−4	−11	60′	59′	44′	−43	−58	26′	−25	12′	5′
V.5	−6	−15	46′	−45	−63	42′	−41	11′	−10	16′	7′
V.6	−8	−19	−28	−40	52′	−51	−60	41′	29′	20′	9′
V.7	−12	−39	47′	32′	−31	−46	−56	−66	57′	40′	13′
V.8	−14	−55	38′	−37	−59	51′	34′	−33	−50	56′	15′
V.9	−16	−22	−34	−52	−57	61′	58′	53′	35′	23′	17′
V.10	−18	−26	−61	66′	50′	−49	64′	63′	62′	27′	19′
V.11	−20	−30	−44	37′	24′	−23	−36	−62	45′	31′	21′
V.12	−29	−42	−64	48′	−47	65′	55′	54′	−53	43′	30′

Vertices of second embedding: N

V.1	−3	−9	−21	−32	−63	64′	33′	22′	10′	4′	1′
V.2	−1	−5	−13	−44	−66	45′	42′	−41	14′	6′	2′
V.3	−2	−7	−17	−24	−38	−48	−39′	25′	18′	8′	3′
V.4	−4	−11	40′	−39	−49	28′	−27	36′	−35	12′	5′
V.5	−6	−15	−62	53′	30′	−29	−52	11′	−10	16′	7′
V.6	−8	−19	−28	−50	−57	66′	58′	51′	29′	20′	9′
V.7	−12	−46	31′	21′	−20	−30	−54	47′	38′	−37	13′
V.8	−14	−60	56′	50′	49′	48′	−47	−55	−64	61′	15′
V.9	−16	−22	−34	−45	57′	−56	−65	46′	35′	23′	17′
V.10	−18	−26	−42	34′	−33	55′	54′	−53	43′	27′	19′
V.11	−23	−36	−43	62′	−61	63′	59′	−58	44′	37′	24′
V.12	−25	−40	52′	−51	−59	32′	−31	65′	60′	41′	26′

Faces of M

F.1	−3	−2	−1	F.16	15′	46′	−56	F.31	33′	−50	−49
F.2	1′	−5	−4	F.17	17′	−24	−23	F.32	34′	−52	−51
F.3	2′	−7	−6	F.18	18′	−26	−25	F.33	35′	−54	−53
F.4	3′	−9	−8	F.19	19′	−28	−27	F.34	37′	−59	44′
F.5	4′	−11	−10	F.20	20′	−30	−29	F.35	38′	−65	55′
F.6	5′	−13	−12	F.21	21′	−32	−31	F.36	39′	47′	65′
F.7	6′	−15	−14	F.22	22′	−34	−33	F.37	40′	52′	−57
F.8	7′	−17	−16	F.23	23′	−36	−35	F.38	42′	−64	63′
F.9	8′	−19	−18	F.24	24′	−38	−37	F.39	43′	−58	53′
F.10	9′	−21	−20	F.25	26′	−61	58′	F.40	45′	−63	62′
F.11	10′	16′	−22	F.26	27′	36′	−62	F.41	48′	49′	64′
F.12	11′	60′	41′	F.27	29′	−42	−41	F.42	50′	56′	−66
F.13	12′	−39	25′	F.28	30′	−44	−43	F.43	51′	−60	59′
F.14	13′	28′	−40	F.29	31′	−46	−45	F.44	57′	61′	66′
F.15	14′	−55	54′	F.30	32′	−48	−47				

Faces of N

F.1	−3	−2	−1	F.16	15′	−62	−61	F.31	31′	65′	46′
F.2	1′	−5	−4	F.17	17′	−24	−23	F.32	32′	−63	59′
F.3	2′	−7	−6	F.18	18′	−26	−25	F.33	33′	55′	−64
F.4	3′	−9	−8	F.19	19′	−28	−27	F.34	34′	−45	42′
F.5	4′	−11	−10	F.20	20′	−30	−29	F.35	38′	−48	−47
F.6	5′	−13	−12	F.21	21′	−32	−31	F.36	39′	−49	48′
F.7	6′	−15	−14	F.22	22′	−34	−33	F.37	43′	62′	53′
F.8	7′	−17	−16	F.23	23′	−36	−35	F.38	44′	−66	58′
F.9	8′	−19	−18	F.24	24′	−38	−37	F.39	45′	57′	66′
F.10	9′	−21	−20	F.25	25′	−40	−39	F.40	47′	−55	54′
F.11	10′	16′	−22	F.26	26′	−42	−41	F.41	50′	−57	−56
F.12	11′	40′	52′	F.27	27′	36′	−43	F.42	51′	−59	−58
F.13	12′	−46	35′	F.28	28′	−50	49′	F.43	56′	−65	60′
F.14	13′	−44	37′	F.29	29′	−52	−51	F.44	61′	63′	64′
F.15	14′	−60	41′	F.30	30′	−54	−53				

Symbol for *M* (listing of its function *p*)

1	7	2	3	3	11	4	5	5	15
6	1	7	19	8	9	9	23	10	2
11	27	12	13	13	31	14	4	15	35
16	17	17	39	18	6	19	43	20	21
21	82	22	8	23	50	24	25	25	79
26	10	27	108	28	29	29	111	30	12
31	20	32	33	33	45	34	14	35	49
36	37	37	53	38	16	39	57	40	41
41	61	42	18	43	65	44	32	45	69
46	47	47	73	48	34	49	77	50	51
51	116	52	36	53	123	54	55	55	26
56	38	57	81	58	59	59	85	60	40
61	89	62	63	63	93	64	42	65	97
66	67	67	101	68	44	69	105	70	71
71	54	72	46	73	88	74	75	75	110
76	48	77	130	78	24	79	113	80	56
81	120	82	83	83	126	84	58	85	106
86	87	87	117	88	60	89	124	90	91
91	30	92	62	93	78	94	95	95	128
96	64	97	96	98	99	99	131	100	66
101	118	102	103	103	80	104	68	105	115
106	107	107	109	108	70	109	129	110	28
111	100	112	92	113	132	114	104	115	121
116	86	117	119	118	74	119	22	120	102
121	114	122	52	123	125	124	72	125	127
126	90	127	98	128	84	129	94	130	76
131	122	132	112						

Symbol for *N* (listing of its function *p*)

1	7	2	3	3	11	4	5	5	15
6	1	7	19	8	9	9	23	10	2
11	27	12	13	13	31	14	4	15	35
16	17	17	39	18	6	19	43	20	21
21	104	22	8	23	70	24	25	25	74
26	10	27	82	28	29	29	121	30	12
31	20	32	33	33	45	34	14	35	49
36	37	37	53	38	16	39	57	40	41
41	61	42	18	43	65	44	32	45	69
46	47	47	73	48	34	49	77	50	51
51	81	52	36	53	85	54	55	55	98
56	38	57	101	58	59	59	105	60	40
61	92	62	63	63	118	64	42	65	127
66	67	67	84	68	44	69	91	70	71
71	54	72	46	73	87	74	75	75	93
76	48	77	96	78	79	79	22	80	50
81	119	82	83	83	89	84	52	85	106
86	72	87	116	88	26	89	132	90	68
91	130	92	24	93	108	94	95	95	97
96	76	97	99	98	78	99	111	100	56
101	115	102	103	103	80	104	58	105	124
106	107	107	109	108	60	109	66	110	94

111	120	112	113	113	90	114	100	115	131
116	117	117	125	118	102	119	129	120	28
121	128	122	123	123	86	124	30	125	122
126	64	127	126	128	110	129	62	130	112
131	114	132	88						

We can observe that the symbols start differing from 21st. coordinates on. So the embeddings are really distincts!

The fact suggests that the problem of minimum embeddings of complete graphs, even in the regular cases, most likely, have many different solutions.

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