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Twistors
Bridges among 3-manifolds

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Abstract

The (bipartite) 3-gems are special edge-colored graphs which induce the (orientable) 3-manifolds. Each 3-manifold is induced in this way and there are simple combinatorial moves on 3-gems which replace topological homeomorphisms. These moves are named *dipole moves*. In this paper a simple configuration on bipartite 3-gems, named *twistor* is isolated. A twistor can be *twisted* in two different ways yielding other two twistors and providing simple moves internal to the class of 3-gems. We prove that by recoupling twistors and by 1- and 2-dipole moves we can transform any bipartite 3-gem into any other. Therefore, the twistors are like brigdes among all orientable 3-manifolds. There is an important connection (not treated here) between this result on these combinatorial twists and the basic theorem of Lickorish that the 3-sphere can be reached from any 3-manifold by removing a finite number of disjoint solid tori and pasting them back differently. This connection will be algorithmically explored elsewhere.

1. Introduction

A *3-gem* can be defined as the dual of a pseudo-triangulation of a closed 3-manifold with a labeling of its 0-cells (vertices) so that each tetrahedra has four differently labeled vertices. More detailed definition is given in the next section.

By a basic Theorem of Ferri and Gagliardi [2], two closed 3-manifolds are homeomorphic iff any 3-gems inducing them are linked by a finite sequence of simple moves called 1- and 2-dipole moves (see Theorem 1). The 3-gems are special kind of edge-colored graphs and because of this, they are very simple to manipulate in a computer. Moreover, they have a rich internal structure and a simplification theory, which have been enabling their topological classification. This was done with success for bipartite 3-gems up to 28 vertices: that is, the homeomorphism problem for the orientable 3-manifolds induced by the bipartite 3-gems with at most 28-vertices has been solved [11,12,10]. This was recently extended for 3-gems with 30 vertices (joint

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work with Cassiano Durand and Said Sidki). The method seems to go furher, limited only by the huge amount of computer time. The 3-gems can also be used to provide concrete computations of the new quantum 3-manifold invariants [5]. The book [6] is a self-contained combinatorial approach to these new invariants; Ch. 13 of this book is a brief exposition on 3-gem theory.

This work intends to close a gap in the theory of 3-gems by providing an internal construction leading from any closed orientable 3-manifold to any such other 3-manifold by means of simple moves on simple configurations specific to 3-gems. Since the moves here presented have inverses of the same type, it is enough to prove that the 3-gem with two vertices, s^3 (which induces S^3), is reachable from any other. Therefore, we define a move on bipartite 3-gems as *universal* if in addition to the 1- and 2-dipole moves they link any bipartite 3-gem to s^3. In this work we isolate a configuration on bipartite 3-gems, a *twistor*, and prove that an associated operation, *twisting a twistor*, is universal.

In getting down to s^3, an adequate choice in the sequence of the twists is guided by a composite complexity function (an ordered triple of positive integers) ordered lexicographically which can be easily obtained from the 3-gems and *measures* them. This complexity is made monotonically decreasing (see Theorem 3) with appropriate twists followed by an operation called *monopole elimination*, which consists of an special composition of 1- and 2-dipole moves. The method of proof is conceptually simple and it uses only first principles on the combinatorics of 3-gems.

To establish the result I introduce the *thin presentation for a bipartite 3-gem*, which is of interest in itself. This presentation provides proofs of results (as the one here presented) and it also simplifies many aspects in the theory of 3-gems.

2. Brief review of 3-gems

An *$(n+1)$-graph* is a finite graph G where at each vertex meet exactly $n+1$ differently colored edges. The total number of colors is also $n+1$. An *m-residue* is a connected component of a subgraph generated by m specified colors. Note that the 2-residues are even-sided bicolored polygons in G, also named *bigons*. Let G^2 be the 2-complex obtained from G by attaching a 2-cell to each 2-residue. A *three-dimensional graph encoded manifold* or simply a *3-gem* is a $(3+1)$-graph satisfying the following condition:

$$\alpha_G = v_G + t_G - b_G = 0,$$

where v_G, t_G, b_G, stand for the number of 0-, 3- and 2-residues of G. The integer α_G is called the *agemality* of the $(3+1)$-graph G, and is in general non-negative [13].

If the G is a 3-gem, i.e., its agemality is null, then each 3-residue with its attached disks form a topological 2-sphere [9]. Let G^3 be the 3-complex obtained from G^2 by attaching a 3-cell to each such 2-sphere. It can be shown that the associated topological space $|G| = |G^3|$ is a closed 3-manifold and that each such 3-manifold arises in this

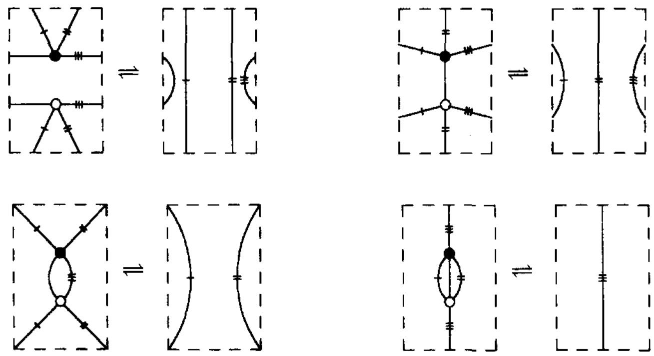

Fig. 1. The dipole moves.

way [13]. If a closed manifold M^3 and a 3-gem G satisfy $M^3 \cong |G|$, then we say that G *induces* M^3.

The orientability of the manifold is apparent from any gem G inducing it: indeed, the manifold is orientable iff G is a *bipartite graph*, i.e., its vertices can be labeled as *black* and *white* so that any edge links a white vertex to a black one (there are no odd cycles (or polygons) in the graph). From the construction for $|G|$ also follows that the interchange of any two colors induces a reversal of orientation and so does the interchange of the black and white classes of vertices.

The colors attached to the edges of a 3-gem are labeled $0, 1, 2, 3$. Two vertices linked by k edges ($k = 0, 1, 2, 3$), whose color set is K constitutes a *k-dipole* if they are in distinct components of the subgraph generated by the colors $\{0, 1, 2, 3\} \backslash K$. The *cancellation of a k-dipole* is the following operation: remove the 2 vertices and all k edges between them; this gives $2 \times (4 - k)$ pendant edges; identify pairwise the pendant ends incident to edges of the same color. The inverse operation is named *k-dipole creation*. A *k-dipole move* is either the cancellation or the creation of the dipole.

Fig. 1 displays the dipole moves for $k = 0, 1, 2, 3$. The colors of the edges are indicated by the number of marks in them. Of course, the moves are indicated up to color permutations.

Note that these are not entirely local moves in the sense that we are assuming the exterior connections to imply a k-dipole. It is not difficult to observe that the k-dipole moves for $k = 1, 2, 3$ do not change the induced manifold, while 0-dipole cancellation is the attachment of a handle [8]. A basic result in the theory of 3-gems is the following Theorem:

Theorem 1 (Sufficiency of 1- and 2-dipole moves, Ferri and Gagliardi [2]). *If* M^3 *and* N^3 *are homeomorphic* 3*-manifolds, then any* 3*-gem* G_M *inducing* M^3 *can be transformed into any* 3*-gem* G_N *inducing* N^3 *by means of a finite sequence of* 2- *and* 1*-dipole moves.*

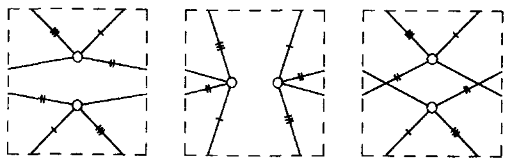

Fig. 2. The three twistors.

In fact, this result is proved in a stronger setting — the crystallization moves — and is proved for all dimensions. A *3-crystallization* is a 3-gem with a minimum number of 3-residues, i.e., four. Any 3-manifold can be induced by a 3-crystallization: just cancell 1-dipoles as long as they are found.

The colors of the edges linking the two vertices of a 1- or 2-dipole are said to be *involved in the corresponding dipole move.*

Lemma 1 (Switching Lemma, Ferri [1]). *Let H_{ij} denote the crystallization obtained from a crystallization H by exchanging two distinct arbitrary colors of their edges. Then H_{ij} is also obtainable from H by a finite sequence of* 1- *and* 2-*dipole moves, each one of these involving color i and not involving color j.*

3. Twistors, antipoles and the main result

An operation T internal to the class of bipartite 3-gems is called *universal* if any 3-gem can be reduced to the 3-gem with 2 vertices (inducing S^3) by means of a finite sequence of T's and 1- and 2-dipole moves. In this section we present two simple configurations inducing universal operations.

A ^{1}*twistor* in a bipartite $(3+1)$-graph is a pair of vertices in the same class which are in the same 01-gon, in the same 23-gon and in distinct 02-, 03-, 12- and 13-gons. To define a ^{2}*twistor* and a ^{3}*twistor* interchange the roles of the pairs of colors $(1, 2)$ and $(1, 3)$, respectively.

Proposition 1 (Twistors). *Consider three partial bipartite* $(3+1)$*-graphs, which differ only locally as shown in Fig. 2. The vertices in the first diagram form a* ^{1}*twistor iff the vertices in the second one form a* ^{2}*twistor iff the vertices in the third one form a* ^{3}*twistor.*

Proof. The proposition is easily checked by the direct inspection of the outside connections of the bigons in the diagrams, when confronted with the definitions of the three types of twistors. Indeed, saying that the first is a ^{1}twistor, or that the second is a ^{2}twistor or that the third is a ^{3}twistor, settles the outside connections in precisely the same way. □

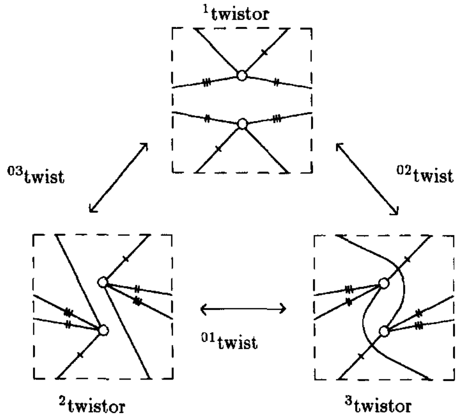

Fig. 3. The structure of twistors and twists.

Let (i,j,k) be a permutation of $(1,2,3)$. The ^{0j}*twist of an* ^{i}*twistor* is the operation of switching the 0-neighbors and the j-neighbors of the two vertices. Note that this produces a ^{k}twistor. These six twistings are easily followed in Fig. 3 (where the twistors are depicted differently from Fig. 2).

Proposition 2. *Twisting twistors is an internal operation in the class of bipartite 3-gems.*

Proof. Note that the number of vertices, of bigons and of 3-residues is the same in any two of the $(3+1)$-graphs of Fig. 2 (or Fig. 3). This is clear for vertices. For bigons it happens because of a rebalance on the various types of ij-gons. For 3-residues, it follows because any two displayed vertices, in the three graphs, are in the same c-missing 3-residue, for $c \in \{0,1,2,3\}$. Since the other 3-residues are unchanged, there is an 1–1 correspondence among the 3-residues of the three $(3+1)$-graphs. Thus, their agemality coincide; if one is a 3-gem so are the others, if one is a 3-crystallization, so are the others. □

There is a dual construction to get the manifold $|G|$ associated to a 3-gem G. Consider a collection of tetrahedra, each with the colors $\{0,1,2,3\}$ labeling its four vertices. These tetrahedra are in 1–1 correspondence with the vertices of G. For each i-colored edge of G we glue the pair of tetrahedra corresponding to its ends via the triangular face not containing i so as to match the other 3 colors, $\{j,k,l\}$. Do this for every edge of G and the result is a manifold $|G|$, if G is a 3-gem [4]. This tridimensional dual complex associated to a 3-gem G is denoted G_d^3. This construction produces the same manifold in the case of a 3-gem G, but it asssociates a topological space to every subgraph of G, contrary to the primal construction, where the space is only defined for the whole G and only in the case that it is a 3-gem. Nevertheless, the graphic nature of the primal model makes it substantially better for computations.

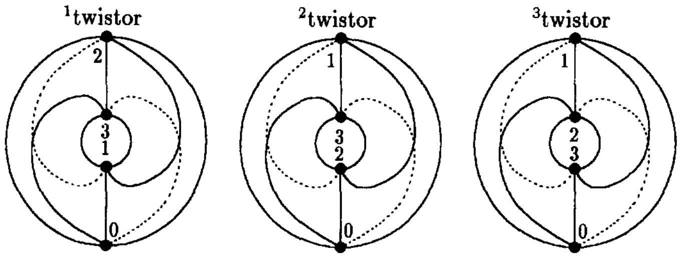

Fig. 4. Dual manifestation of twistors.

Both constructions are important for the topological interpretation of the fundamental objects and operations in the theory of 3-gems. In fact, it is convenient to consider the primal and the dual complexes at hand. Observe the following correspondence between dual cells:

- a vertex in v in $G \rightleftharpoons$ a solid tetrahedron T_v in G_d^3 whose vertices are labeled $0,1,2,3$;
- an i-colored edge e_i in $G \rightleftharpoons$ a triangular 2-cell E_i in G_d^3 whose vertices are labeled with the 3 colors distinct from i;
- a bigon B_{ij} using colors i,j in $G \rightleftharpoons$ an edge b_{ij} in G_d^3 whose ends are labeled h,k, where (h,i,j,k) is a permutation of $(0,1,2,3)$;
- a 3-residue V_i in G not containing color $i \rightleftharpoons$ a vertex of G_d^3 labeled i.

We present the dual manifestation of the twistors in Fig. 4. Under geometric duality the three twistors correspond, respectively, to the following configurations formed by two tetrahedra with $0,1,2,3$ labeling their vertices, embedded in R^3 with the same orientation and having precisely one pair of edges in common.

Theorem 2 (Main Result). *The twisting of twistors is a universal operation on bipartite* 3*-gems.*

The proof of this theorem is postponed to Section 6. From the dual interpretation this result implies the fundamental theorem of Lickorish that any orientable 3-manifold can be obtained from S^3 by removing a disjoint set of solid tori and pasting them back differently [7]. Algorithmic consequences of this implication will be considered elsewhere.

In proving Theorem 2 we must use the two types of twists on ^{1}twistors. However, from Ferri's Switching Lemma [1] stated as Lemma 1, it is possible to interchange any two colors by 1- and 2-dipole moves. Therefore, we can be very specific.

Corollary 1. *The* ^{03}*twist of a* ^{1}*twistor is a universal operation on* 3*-gems.*

Another configuration, needed in the proof of the main result, which is closely related to the twistor is the *antipole*. It also appears in three forms named ^{1}*antipole*,

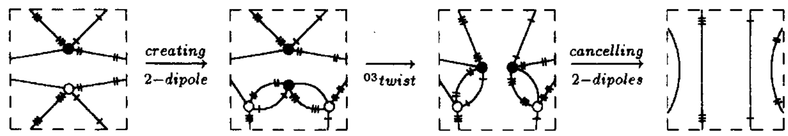

Fig. 5. Twisting of a twistor implies cancellation of an antipole.

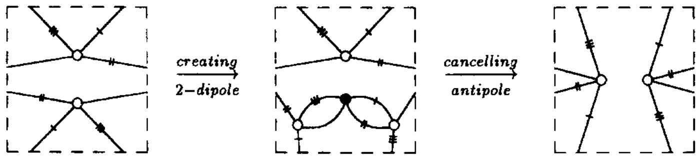

Fig. 6. Cancellation of antipole implies twisting of a twistor.

^{2}*antipole*, ^{3}*antipole*. Their definitions differ from the ones of the twistors in that the pair of vertices are in distinct classes. *Cancelling an antipole* is the operation similar to the cancellation of a 0-dipole (see Fig. 1): the two vertices are removed and the pair of ends of the same colors are identified.

Under the presence of 2-dipole moves, antipole cancellation is equivalent to twisting a twistor, as the next two propositions show.

Proposition 3. *Cancellation of an antipole is factorable as a 2-dipole creation, a twist and two 2-dipole cancellations.*

Proof. Consider the passages as shown in Fig. 5.

Assume the first configuration is a ^{1}antipole. Then, in the second, the black vertices form a ^{1}twistor. Applying a ^{03}-twist to it produces two 2-dipoles whose cancellation completes the factorization. The treatment of the other two types of antipoles cancellations are analogous. □

Proposition 4. *Twisting a twistor is factorable as a 2-dipole creation followed by an antipole cancellation.*

Proof. Consider the passages as shown in Fig. 6.

Let the first configuration be a ^{1}twistor. Then, in the second, the pair of separated vertices in distinct classes form a ^{1}antipole and the third configuration is a ^{2}twistor. The factorizations of the other five types of twistings of twistors are similar. □

Observe that the inverse of a ^{0i}twist is also a ^{0i}twist. It is interesting to observe that under the presence of 2-dipole moves, the inverse of operation of antipole cancellation is accomplished by the same type of operation.

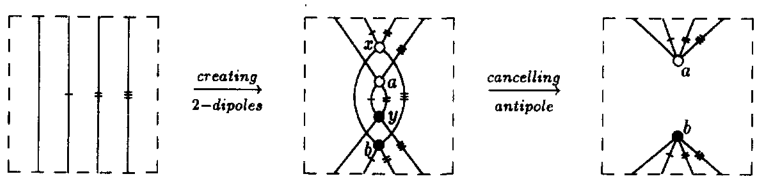

Fig. 7. Cancellation of antipole implies its inverse.

Proposition 5. *The inverse of an antipole cancellation is factorable as a two 2-dipole creations followed by an antipole cancellation.*

Proof. Suppose that in the left figure of Fig. 7 the edges of colors 0 and 1, 1 and 2, 2 and 3 and 3 and 0 are in the same bigon. Suppose also that they are part of a bipartite 3-gem and their upper ends are •vertices so that their lower ends are °vertices. Then in the first passage we create a 2-dipole with vertices a and y and a 2-dipole with vertices x and b. Observe that, in the middle figure, vertices x and y form a ^{1}antipole whose cancellation produces the third one. In it, vertices a and b constitute a ^{2}antipole. The cancellation of this antipole produces the initial figure and, therefore, the inverse of a ^{2}antipole cancellation is accomplished by two 2-dipole creations and an ^{1}antipole cancellation. □

Corollary 2. *Antipole cancellation is a universal operation on bipartite 3-gems.*

Proof. This follows from Theorem 2 and Proposition 4. □

4. 23-complexity and ρ_{23}-moves

In the proof of Theorem 2 we need to evaluate the '*size*' of a 3-gem. The most natural measure would be the number of its vertices. However, an apparently deep obstruction exists to prevent such a natural evaluation work: there is a rich but incomplete theory, which resembles the one for the 4-color Theorem. We are then led to the following '*measure*'. The *23-complexity* of a 4-graph G is

$$\xi_{23}(G)=(b_{23}(G),m_{23}(G),a_2(G)),$$

where $b_{23}(G)$ is the number of 23-gons in G, m_{23} is the number of 2-colored edges in a 23-gon with a minimum number of edges and $a_2(G)$ is the total number of 2-colored edges, i.e., $\frac{1}{2}v_G$. Observe that the definition is bias towards the pair of colors 2, 3. Of course, this is only one of six possible choices.

The 23-complexities are ordered lexicographically. The guiding idea in the proof of Theorem 2 is to define composite moves formed by twists and dipole moves so that each application of these moves decreases ξ_{23}. So, we arrive to the 2-vertex 3-gem s^3, having the smallest possible ξ_{23}-complexity $(1,1,1)$.

The simplest ξ_{23}-reducing moves are the 1- and 2-dipole cancellations:

Lemma 2. *Let G be a 3-gem with a 1-dipole or a 2-dipole. Cancellation of the dipole produces a 3-gem G′ with* $\xi_{23}(G') < \xi_{23}(G)$.

Proof. Suppose we have cancelled a 1-dipole. If its color is 0 or 1 then $b_{23}(G)$ drops by one. If the 1-dipole is of color 2 or 3, then $b_{23}(G') = b_{23}(G)$, $m_{23}(G') \leqslant m_{23}(G)$, but $a_2(G') < a_2(G)$ thereby proving that $\xi_{23}(G') < \xi_{23}(G)$.

The case of 2-dipole is also easy: if its edges are of color $(0,1)$, or $(2,3)$, then $b_{23}(G)$ drops by one. In the other four cases, as before, $b_{23}(G') = b_{23}(G)$, $m_{23}(G') \leqslant m_{23}(G)$, but $a_2(G') < a_2(G)$. So, in all cases, $\xi_{23}(G') < \xi_{23}(G)$. □

From this lemma we can restrict attention to 3-crystallizations free of 2-dipoles. One of the ξ_{23}-reducing composite moves that we need is the ρ_{23}-*move* which we start discussing.

Given a bipartite $(3+1)$-graph G, a ρ_h-*pair* $(2 \leqslant h \leqslant 3)$ is a pair of edges $\{a_1, a_2\}$ equally colored which is contained in h bigons. The *switching* of a ρ_h-pair is the passage from G to G' obtained by replacing $\{a_1, a_2\}$ by new edges $\{a'_1, a'_2\}$ having the same ends and preserving the bipartition.

The two lemmas below appear in [3], in another context and with a slightly different terminology. Complete proofs are also given in [10].

Lemma 3. *Let G be a 3-crystallization and (h,i,j,k) a permutation of its edge colors $(0,1,2,3)$. Suppose that $\{a_1, a_2\}$ is a ρ_2-pair of color h appearing together in the same hi-gon, in the same hj-gon and belonging to distinct hk-gons. Let G′ be the resulting $(3+1)$-graph after switching the pair. Then G′ is a connected bipartite 3-gem, $|G'| \cong |G|$, $t_k(G') = 2$ and $t_c(G') = 1$ for $c \neq k$ (see Fig. 8).*

Proof. A ρ_2-pair switching is factorable as a 1-dipole creation followed by a 2-dipole cancellation; see Lemma 8 of [3], or Lemma 4 of [10]. □

Lemma 4. *Let G be a bipartite 3-crystallization, (h,i,j,k) a permutation of $(0,1,2,3)$, $\{a_1, a_2\}$ a ρ_3-pair of color k and G′ the resulting $(3+1)$-graph after switching the*

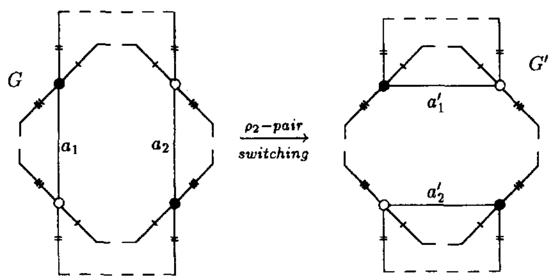

Fig. 8. Switching a ρ_2-pair.

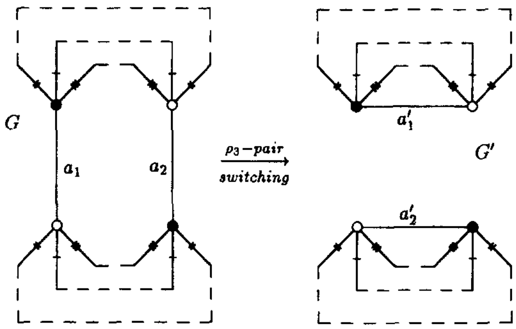

Fig. 9. Switching a ρ_3-pair.

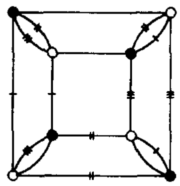

Fig. 10. $s^1 \times s^2$, canonical 3-gem inducing $S^1 \times S^2$.

pair. Then G' is a connected 3-gem, $|G| \cong |G'|\#(S^1 \times S^2)$, $t_h(G')=1$ and $t_c(G')=2$, for $c \neq h$ (see Fig. 9).

Proof. A ρ_3-pair switching is factorable as a 0-dipole creation followed by a 1-dipole cancellation; see Lemma 9 of [3] or Proposition 20 of [10]. □

The proof of the above lemma involves a 0-dipole creation. In the present context this is not directly usable. The following lemma is needed to fix the situation.

Lemma 5. *A ρ_3-pair switching is factorable as one twist preceded and followed by a finite number of 1- and 2-dipole moves.*

Proof. Let G and G' be the ones of the preceding lemma. Consider the canonical 3-gem for $S^1 \times S^2$ [8], naming it $s^1 \times s^2$ as shown in Fig. 10. Let x be any vertex of G' and y be any external vertex of $s^1 \times s^2$. Denote by $G'^x\#^y s^1 \times s^2$ the resulting 3-gem obtained by cancelling the 0-dipole x, y in the 2-component 3-gem $G' \cup s^1 \times s^2$. By a sequence of 1- and 2-dipole moves G can be transformed into $G'^x\#^y s^1 \times s^2$. This follows (non-constructively) from the Ferri–Gagliardi Theorem, since these 3-gems induce homeomorphic 3-manifolds. In fact, we can present explicitly a sequence realizing the passage — see the Walking Lemma of [8] and Theorem 2 of the same article.

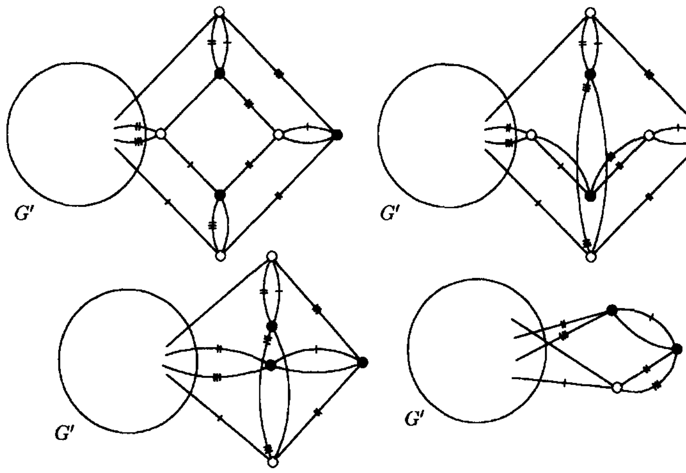

Fig. 11. A 3-gem equivalent to G by 1- and 2-dipole moves.

Note that the black pair of opposite vertices in the internal square of $s^1 \times s^2$ is a ^{1}twistor. Since y is external, this twistor is present in $G'^x \#^y s^1 \times s^2$. By applying ^{03}-twist to it produces three 2-dipoles. By cancelling these dipoles we obtain G'. In Fig. 11, the final cancellation of a 2-dipole, which yields G' back, is not shown. □

A *ρ-pair* is either a ρ_2- or a ρ_3-pair. Once a ρ-pair is present in a 3-crystallization we can get another by a *ρ-move*: (a) switching the ρ-pair; (b) cancellation of one 1-dipole, in case of a ρ_2-pair (see Lemma 2) or the cancellation of three 1-dipoles, in case of a ρ_3-pair (see Lemma 3). Thus, the ρ-move is internal to 3-crystallizations and the resulting graph has less vertices. Moreover, the induced spaces are the same up to connected sum with $S^1 \times S^2$. Note however that a ρ-move can increase the 23-complexity. Indeed, switchings a ρ_2-pair may increase the number of 23-gons and the subsequent 1-dipole (of color 0 or 1) cancellation may increase the size of a minimum 23-gon. We are led to the following definition. A *ρ_{23}-pair* is one of the following:

- a ρ_3-pair;
- a ρ_2-pair of color 0 or 1;
- a ρ_2-pair of color 2 or 3 lying in a minimum 23-gon;

A *ρ_{23}-move* is a ρ-move applied to a ρ_{23}-pair.

Lemma 6. *A ρ_{23}-move applied to a bipartite crystallization decreases its 23-complexity.*

Proof. If the pair is a ρ_3-pair or a ρ_2-pair of colors 0 or 1, then its switching does not change ξ_{23}. As the cancellations of 1-dipoles decrease it, we are done.

If the pair is a ρ_3-pair of color 2 or 3, then its switching increases by one the number of 23-gons. However, two among the three cancellations of 1-dipoles, are on 1-dipoles of colors 0 and 1 (see Lemma 3). These cancellations drops by two the number of 23-gons.

If the pair is a ρ_2-pair of color 2 or 3, then its switching may decrease or increase by one the number of 23-gons. If it decreases, we are done. If it increases follows that all the available 1-dipoles must be of color 0 or of color 1: if $\{h,k\}=\{2,3\}$ in Lemma 2, then b_{23} would decrease, contrary to our hypothesis. Therefore, cancellation of the 1-dipole brings b_{23} to its previous value. The final crystallization G' has the same number of 23-gons as the original G but it has a smaller minimum 23-gon: a minimum 23-gon β of G has been subdivided into two and with the 1-dipole cancellation at least one of these reduced 23-gon remains or else they coalesce into a 23-gon of G' having two less edges than β. □

A bipartite 3-crystallization without 2-dipoles and free of ρ_{23}-pairs is called a ρ_{23}-*reduced* 3-*gem*. The above discussion enable us to focus on such restricted class of bipartite 3-gems.

5. Monopoles and a thin presentation for 3-gems

On the way to the proof of the Theorem 2 we isolate another configuration and an associated operation which decreases ξ_{23}.

A vertex u of a $(3+1)$-graph is a ^{1}*monopole* if the 01-gon and the 23-gon containing u have only this vertex in common. As before there are ^{2}*monopoles* and ^{3}*monopole* obtained by interchanging the roles of colors $(1,2)$ and $(1,3)$, respectively.

Lemma 7. *If a* ^{1}*monopole is present in a* 3-*gem* G, *then, by a sequence of* 2-*dipole and* 1-*dipole moves,* G *is transformable into a* 3-*gem* G' *so that* $\xi_{23}(G') < \xi_{23}(G)$.

The final 3-gem G' might have much more vertices than G; however it will have one less 23-gon than it. Since it is the only place where the number of vertices increases, it would be important to find a proof of Theorem 2 which replaces the use of this lemma by some scheme which does not increase the number of vertices.

Before we establish Lemma 7, let us introduce a *thin presentation for* 3-*gems*, which will make its proof transparent. It also simplifies the understanding of the rest of arguments and many other proofs in the theory.

To get the *thin presentation of a* 3-*gem* start with an embedding of its 0-missing 3-residues in the plane. The 0-colored edges may have crossings and their embeddings are not important. In this context they are called *wires*. The *frame* of the thin presentation is a partial dual formed as follows: consider a black vertex inside each 12-gon and a white one inside each 13-gon. For each original 1-colored α edge consider a dual edge crossing it transversally once and linking the newly put vertices inside the 12-gon and

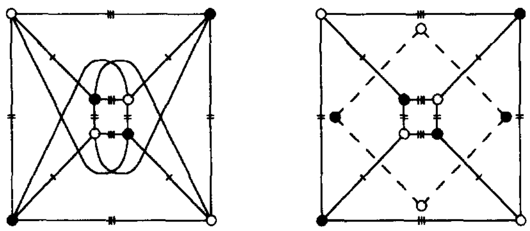

Fig. 12. Initial steps in thinning the presentation of 3-gem rp^3.

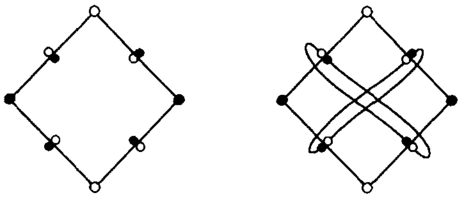

Fig. 13. Final steps in thinning the presentation of 3-gem rp^3.

the 13-gon containing α. The embedding of the frame, induced by the embedding of the 0-missing residues is an information to be kept. A simple example, from the 3-gem rp^3, inducing RP^3 is given in Fig. 12, where the frame appears in dashed lines.

To conclude, contract each original 1-colored edge so that the small circles around their ends touch forming a *dumb-bell* and put back the wires. The result of the above example is Fig. 13.

The alternating circular elements formed by wires and dumb-bells are called *wired cycles*. They are in a clear 1–1 correspondence with the original 01-gons. The *thin presentation* is the union of the frame (for which the embedding is important) and the wired cycles. There is an 1–1 correspondence between various other elements of 3-gems G and of their thin presentations $G^\star$. For instance:

- a black/white vertex of $G \rightleftharpoons$ half of a dumb-bell of the same color of $G^\star$;
- a 0-colored edge of $G \rightleftharpoons$ a wire in $G^\star$;
- a 1-colored edge of $G \rightleftharpoons$ a dumb-bell in $G^\star$;
- a 2-colored edge of $G \rightleftharpoons$ an angle between two edges of a black vertex of $G^\star$;
- a 3-colored edge in $G \rightleftharpoons$ an angle between two edges of a white vertex of $G^\star$;
- a 01-gon of $G \rightleftharpoons$ a wired cycle of $G^\star$;
- a 12-gon of $G \rightleftharpoons$ a black vertex of $G^\star$;
- a 13-gon of $G \rightleftharpoons$ a white vertex of $G^\star$;
- a 23-gon of $G \rightleftharpoons$ a face in the frame of $G^\star$;
- a 1 missing 3-residue of $G \rightleftharpoons$ a component of the frame of $G^\star$;

Observe that the construction is reversible and G is recoverable from $G^\star$. The embeddings of the wires are not relevant because only the combinatorial information between the matching of the sides of the edges in the frame is important to recover

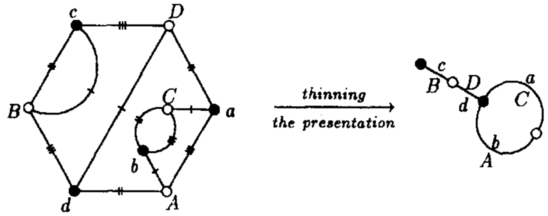

Fig. 14. Combinatorial thin presentation: the wires and dumb-bells are indicated implicitly by matching the sides of the edges.

the 0-colored edges and, thus, the whole 3-gem from its thin presentation. An example (inducing $S^1 \times S^2$) of such a combinatorial thin presentation is shown in Fig. 14. The 0-colored edges and wires are implicitly given by a pair of corresponding lower–upper case letter.

The above example has an *isthmus in the frame*, i.e., an edge whose deletion increases the number of components of the graph. It is easy to see that an isthmus in the frame implies a ρ-pair in the 3-gem: indeed, the 2-colored edges corresponding to the angles at a black vertex formed by the isthmus and another edge are in the same 23-gon and in the same 12-gon. A *chord in a thin presentation* is a wire linking half dumb-bells in the same face of the frame. Note that a chord also induces a ρ-pair: the 2-colored edges incident to the vertices corresponding to the half dumb-bells incident to the chord are in the same 23- and in the same 02-gons. A *minimum face* of a frame is one which has a minimum number of vertices around its boundary (an isthmus is counted twice). It corresponds to a minimum 23-gon.

Proposition 6. *A 3-gem whose frame has an isthmus or a chord in a minimum face is not ρ_{23}-reduced.*

Proof. In each case the induced ρ-pair is a ρ_{23}-pair since their elements are in a minimum 23-gon. The proposition follows from Lemma 5. □

In a thin presentation the manifestation of the dipole moves is very simple. Observe that a 2-dipole in colors $(0,2)$ or $(0,3)$ corresponds to an angle in which the half dumb-bells are directly linked by a wire, or *trivial angle*. Cancelling/creating a 2-dipole involving such colors corresponds to the closing/opening of the associated trivial angle. Note that the identified vertices in Fig. 15 are indeed distinct.

A 2-dipole involving colors $(0,1)$ corresponds to an edge in the frame whose dumb-bell have their half directly linked by a wire, forming a wired cycle with just one wire and dumb-bell, a *trivial wired cycle*. In the case of a 2-dipole, the corresponding edge in the frame is not an isthmus. Cancellation/creation of such a dipole corresponds to deletion/insertion of such an edge with the associated wired cycle as shown in Fig. 16.

Fig. 15. Opening/closing trivial angles.

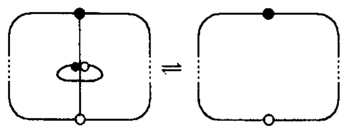

Fig. 16. Deletion/insertion of a trivial wired cycle.

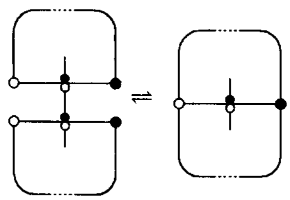

Fig. 17. Pasting/cutting two components.

Proposition 7. *A 3-gem whose thin presentation has a trivial angle or a trivial wired cycle is not ρ_{23}-reduced.*

Proof. A trivial angle corresponds to a 2-dipole. A trivial wired cycle corresponds to a 2-dipole or else the 2-colored edges incident to the half dumb-bells in the cycle form a ρ_3-pair, which is a ρ_{23}-pair, and the result follows from properties established in the last section.

A 1-dipole of color 0 becomes a wire linking two distinct components in the frame of the thin presentation. Cancellating/creating this 1-dipole corresponds to pasting/cutting the components along edges/edge of the frame as shown in Fig. 17.

In the context of thin presentations, an original vertex is a half dumb-bell and is considered as a *meeting* between the face F where the half dumb-bell is inside and the wired cycle containing it. It is easy to see that single meetings between a face F and a wired cycle W correspond to ^{1}monopoles and double meetings between them are candidates for ^{1}antipoles of ^{1}twistors. Indeed, if there are no ρ_{23}-pairs, the candidates qualify.

Proof of Lemma 7. A ^{1}monopole h corresponds to a half dumb-bell which is a single meeting between a face F and wired cycle W. We can decrease the size of W, reproducing this situation, as follows. We start at the black vertex incident to the edge

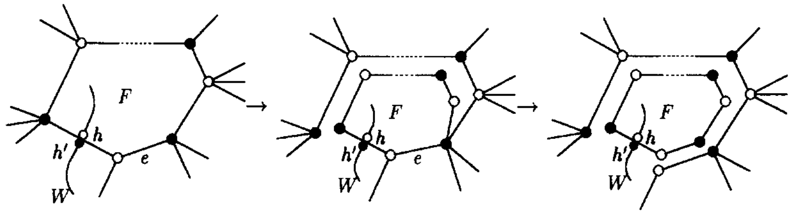

Fig. 18. Monopole h and h-avoiding cut along its face F.

associated to h to open trivial angles until the face F is connected to the rest of the frame by a single edge e. These are 2-dipole creations. Next we do a cutting at e, which is a 1-dipole creation. Observe that in opening angles and in the cutting we did not increase the size of W, since h was the unique meeting between F and W. Now restore the connectivity of the frame by pasting F back using the wire incident to the half dumb-bell h', which with h form a dumb-bell. The situation is reproduced with h being the only meeting between F and W, but now W is of smaller size, since it has lost a dumb-bell and a wire (see Fig. 18).

Iteration of the above process produces a trivial wired cycle, which corresponds indeed to a 2-dipole involving colors $(0,1)$: otherwise the edge of h would be an isthmus and h could not be a ^{1}monopole (the corresponding half dumb-bell h' would be another meeting between F and W). Cancellation of this 2-dipole produces a 3-gem G' which has one less 23-gons than G. □

The process of getting down one 23-gon by means of the above composition of dipole moves, once a ^{1}monopole z is present, is called the *elimination of monopole* z.

A *23-reduced 3-gem* is a ρ_{23}-reduced one which is free of ^{1}monopoles. From the Lemma above established we may restrict ourselves to this class of 3-gems in the proof of Theorem 2.

6. Proof of the main result

The following technical result is the crucial point in establishing Theorem 2.

Theorem 3. *If G has more than 2 vertices and is a 23-reduced 3-gem, then there exists one of the three compositions:*

1. *a ^{03}-twist on a ^{1}twistor followed by a ^{1}monopole elimination,*
2. *a ^{02}-twist on a ^{1}twistor followed by a ^{1}monopole elimination,*
3. *a ^{1}antipole cancellation followed by a ^{1}monopole elimination,*

which transforms G into a bipartite 3-gem G' with $\xi_{23}(G') < \xi_{23}(G)$.

Proof of Theorem 2 (assuming Theorem 3). If G has two vertices, there is nothing to do. Otherwise, get G' from G as given by the above theorem. From G' we can get

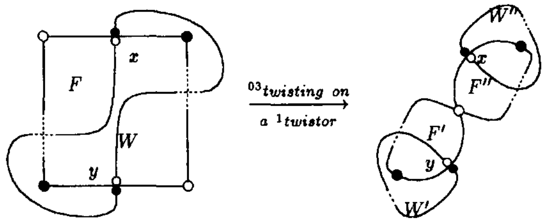

Fig. 19. ^{03}Twist on a w-successive ^{1}twistor (x, y) implies monopole y.

a 23-reduced 3-gem G'' with $\xi_{23}(G'') \leqslant \xi_{23}(G')$ by methods explained in the previous sections. Since $\xi_{23}(G'') < \xi_{23}(G)$ we let G'' take the role of G and iterate the process, until we get the 3-gem with two vertices. Since the ^{1}antipole cancellation is factorable as dipole moves and ^{0i}-twist, it follows that we get down to the 2-vertex 3-gem, by 1- and 2-dipole moves and ^{0j}twistings $(j = 2, 3)$ on ^{1}twistors, proving Theorem 2. □

Proof of Corollary 1. The cancellation of an ^{1}antipole is factorized by ^{03}twisting of a ^{1}twistor (see Fig. 5) and 2-dipole moves. By interchanging, with 1- and 2-dipole moves, the colors 2 and 3 with Ferri's Switching Lemma [1], the ^{02}twisting of a ^{1}twistor becomes the ^{03}twisting of a ^{1}twistor. □

It remains to prove Theorem 3. Its proof follows from three facts, which we isolate as lemmas. The other end of the half dumb-bell x is denoted x'. A double meeting (x, y, F, W) between a face F of the frame and a wired cycle W is called *w-successive* if in the partial wired cycle that goes from x to y avoiding x' there are no meetings between F and W.

Lemma 8. *If G has a w-successive double meeting (x, y, F, W) in a minimum face F of its thin presentation and (x, y) have the same color, it forms a ^{1}twistor and its ^{03}-twisting or its ^{02}-twisting produce 3-gems with a ^{1}monopole whose elimination yields 3-gems G' and G'' satisfying $\xi_{23}(G') < \xi_{23}(G)$ and $\xi_{23}(G'') < \xi_{23}(G)$.*

Proof. Since there are no ρ_{23}-pairs, the fact that (x, y) have the same color implies that (x, y) are the vertices of an ^{1}twistor. The ^{03}-twisting of (x, y) produces a ^{1}monopole y which is the single meeting between F' and W', as shown in Fig. 19.

In the same way, the ^{02}-twisting of (x, y) produces a ^{1}monopole y which is the single meeting between F' and W', as shown in Fig. 20.

These facts follows because the part of the wired cycle W which goes from x to y avoiding x' does not meet F. In the two cases, the elimination of the ^{1}monopole y gives a 3-gem G' having the same number of 23-gons as G but with a smaller minimum face, since F'' remains intact because W' does not meet F''. □

The flexibility in having the two alternative twistings in the previous lemma is crucial in proving the next one.

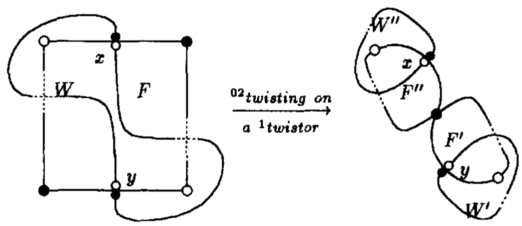

Fig. 20. ^{02}Twist on a w-successive ^{1}twistor (x, y) implies monopole y.

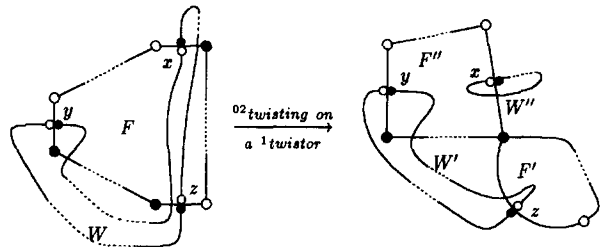

Fig. 21. One possibility for x, y, z.

Lemma 9. *Let F be a minimum face of a thin presentation of a 23-reduced 3-gem G. If there is a wired cycle W which meets F more than twice, then by a ^{02}-twisting or by a ^{03}-twisting followed by a ^{1}monopole elimination we get a 3-gem G' satisfying $\xi_{23}(G') < \xi_{23}(G)$.*

Proof. Let x, y, z be successive meetings between F and W along W. x and y have distinct colors, otherwise the previous lemma applies. The same for y and z. Therefore, we may suppose that x and z are white and y is black. It follows that (x,z) is a ^{1}twistor. Beginning at the black end of the edge of x and using first this edge in a walk around the boundary of F we may find first y and then z or first z and then y. In the first possibility we apply a ^{02}-twist to (y,z), and in the second we apply ^{03}-twist to the same pair. The first situation is as shown in Fig. 21 (up to reversal of orientation).

The other situation is as shown in Fig. 22. In both cases the wired cycle W' meets F' only at z and meets F'' only at y. Elimination of the ^{1}monopole z produces a 3-gem G' which has the same number of 23-gons as G but which have a face F''', having two less edges than F, a minimum face of G. Indeed, at some stage in the cancellation of z, the faces F' and F'' are put together along the edges of y and z. Next in the process, the identified edge is removed, and a face F''' is formed. See the proof of Lemma 7. Since G' satisfies $\xi_{23}(G') < \xi_{23}(G)$, we are done. □

Lemma 10. *Assume that the minimum face F of a thin presentation of a 23-reduced 3-gem G is visited exactly twice by each wired cycle W which meets it. Moreover, assume that these double meeetings form ^{1}antipoles. Then, by a ^{1}antipole cancellation*

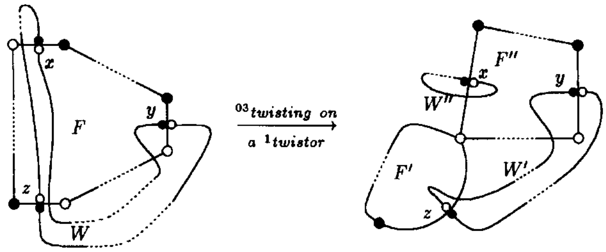

Fig. 22. The other possibility for x, y, z.

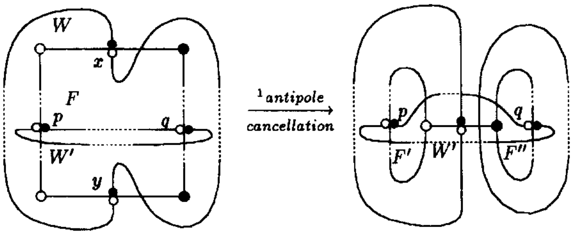

Fig. 23. Antipole cancellation in a balanced situation.

followed by the elimination of a ^{1}*monopole we get a* 3*-gem* G' *satisfying* $\xi_{23}(G') < \xi_{23}(G)$.

Proof. Let W and W' be wired cycles so that their meetings with F alternate around the boundary of F. There exists such a pair otherwise some wired cycle would have their two meetings in adjacent edges of F. This angle would correspond to a trivial angle or the two 0-colored edges incident to the half dumb-bells of the angle would form a ρ_{23}-pair; contradictions in both alternatives.

Let x and y be the meetings of W and F and let p and q be the meetings of F and W'. The cancellation of ^{1}antipole (x, y) is depicted in Fig. 23.

Observe that after the ^{1}antipole cancellation, p is the unique meeting between F' and W'. Thus it is a ^{1}monopole. The elimination of this ^{1}monopole produces 3-gem G'. In this cancellation faces F' and F'' are pasted together along edges containing p and q. The identified edge is deleted forming a face F''' with two less edges than F. No further increases in the number of edges of F''' occur, because W' meets F'' only at q.

Therefore, the final 3-gem G' has the same number of 23-gons as G but its thin presentation has face F'''. It follows that $\xi_{23}(G') < \xi_{23}(G)$, establishing the lemma. □

With everything in place it is easy to prove what is missing.

Proof of Theorem 3. The 3-gem G has no ^{1}monopole, so, every one of its half dumb-bells is part of a double meeting with some wired cycle. If there is a w-successive

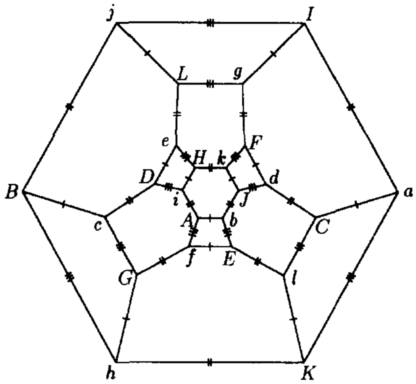

Fig. 24. The superattractor $s^1 \times s^1 \times s^1$ for $S^1 \times S^1 \times S^1$.

double meeting, the pair must form a ^{1}antipole, otherwise we apply Lemma 8. If some wired cycle W meets F more than twice, then apply Lemma 9. Now remains the situation of Lemma 10. By applying it we are done. □

Assume that G is the unique 3-gem having the smallest number of vertices among all that induces M^3. Then G is called a *superattractor* for M^3 [10]. Fig. 24 shows the 3-gem superattractor for the 3-torus. The edges of color 0 are implicitly given by a pair of upper/lower case equal letters.

The above 3-gem, $s^1 \times s^1 \times s^1$, shows that the hypotheses of Lemma 10 are consistent because it has no twistors, only antipoles appearing exactly twice at each face. Therefore, the necessity of using antipole cancellation sometimes do appear. The problem with the factorization of this operation by means of a twist is that the number of vertices increases. We think a monotonically vertex decreasing theory could be available. It would necessarily reveal many deep properties of 3-gems yet unknown.

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