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A homological solution for the Gauss code problem in arbitrary surfaces

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Abstract

Let $\overline{P}$ be a sequence of length $2n$ in which each element of $\{1, 2, \ldots, n\}$ occurs twice. Let P' be a closed curve in a closed surface S having n points of simple self-intersections, inducing a 4-regular graph embedded in S which is 2-face colorable. If the sequence of auto-intersections along P' is given by $\overline{P}$, we say that P' is a 2-face colorable solution for the Gauss code $\overline{P}$ on surface S or a lacet for $\overline{P}$ on S. In this paper we show (by using surface homology theory mod 2), that the set of lacets for $\overline{P}$ on S are in 1–1 correspondence with the tight solutions of a system of quadratic equations over the Galois field $GF(2)$. If S is the 2-sphere, the projective plane or the Klein bottle, the corresponding quadratic systems are equivalent to linear ones. In consequence, algorithmic characterizations for the existence of solutions on these surfaces are available. For the two first surfaces this produces simple proofs of known results. The algorithmic characterization for the existence of solutions on the Klein bottle is new. We provide a polynomial algorithm to resolve the issue.

Keywords: Gauss code problem; Lacets; Closed surfaces; 4-Regular graphs; Medial maps (of graphs on surfaces); Face colorability

1. Introduction

By a *polygon in a graph G* we mean a connected non-null subgraph whose vertices have valence 2. In this work, by a *cycle in a graph* we mean a subset of edges inducing a subgraph

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(possibly empty or non-connected) having only vertices of even valence. This terminology is non-standard in graph theory. However, it seems appropriate here because our cycles are closed 1-chains in the sense of homology mod 2, see Giblin [4]. The standard definition of cycle, as the edge-set of a polygon, is just a particular case of our cycles. The set of cycles form a vectorial space over $GF(2)$ where the sum is given by the symmetric difference. We denote this vector space by $CS(G)$.

Basic concepts on graph theory are given in the book of Bondy and Murty [1]. For topological graph theory see the book of Bonnington and Little [2]. Background material on the topic of graphs embedded on surfaces can be found in the book of Mohar and Thomassen [11]. For background in algebraic graph theory see the book of Godsil and Royle [6].

A *topological map*, or simply a *t-map* $M^t = (G, S)$ is an embedding of a graph G into a closed surface S such that $S\backslash G$ is a collection of disjoint open disks, called *faces*. A *Gauss code* $\overline{P}$ is a cyclic sequence in the set of labels $E = \{1, 2, \ldots, n\}$ in which each $x \in E$ occurs twice. Let P' be a closed curve in a closed surface S having n points of simple self-intersections, inducing a 4-regular graph embedded into S such that the cyclic sequence of self-intersections reproduces $\overline{P}$. If the embedding of P' produces a 2-face colorable t-map, we say that P' is a *lacet* for $\overline{P}$ on S. Without the 2-face colorability condition the algebra derived from maps with a single zigzag [7] is not available and an entirely different problem arises. A solution for the non-2-face colorable case for the projective plane has been given in [12]. Here, however, lacets are 2-colorable. In this case, P' is the *medial map* [6] of a map M formed by a graph G_M embedded into S. M has a single zigzag [8]. The dual of M is denoted D and its phial [8] is denoted P. G_P, the graph of P, has a single vertex (corresponding to the single zigzag). Previous work on the Gauss code problem can be found in [7,9,10,13,14,16]. The last two works solve the 2-face colorable problem for the case of the projective plane. The previous works deal with the planar case in which the 2-face colorability is granted. In the present work we algorithmically solve the problem for the Klein bottle.

The problem for surfaces of Euler characteristic 0 has been considered by in [3]. This paper introduces the terminology *lacet* and finds characterizations for the realization in the torus (Theorem 19) and in the Klein bottle (Theorem 20) in terms of the existence of a pair of 0–1 vectors with certain properties. However, the verification of the existence of these vectors is left undiscussed. In fact, to verify their existence leads to an exponential number of trials. So the theorems do not provide polynomial algorithms for the existence of a lacet on those surfaces. In this algorithmic sense the problems are not solved in [3].

In this paper we obtain an algorithmic solution for the Klein bottle. We also provide a way for finding the surface with the greatest Euler characteristic realizing a Gauss code as a lacet in terms of deciding whether or not a quadratic system of equations over $GF(2)$ is consistent.

The paper is organized as follows. Section 2 briefly reviews the theory of combinatorial maps as given in [8] to state the *Parity Theorem* [7], which is needed in this work. In Section 3 we discuss two linear transformations c_P and $c_{P^\sim}$ arising from a t-map with one zigzag. These function play a central role in our methodology. In Section 4 we state and prove the main results on arbitrary surfaces—orientable and non-orientable. Section 5 we produce the equivalent linear systems for the case of 2-sphere, projective plane and Klein bottle. Finally, short Section 6 consists of a concluding remark and acknowledgments.

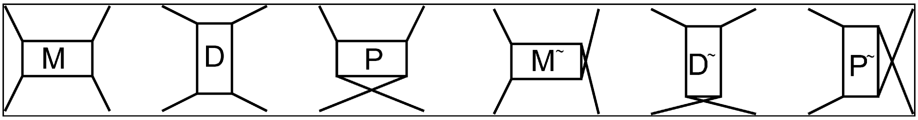

Fig. 1. The six neighborhoods of a rectangle inducing M, D, P, $M^{\sim}$, $D^{\sim}$, $P^{\sim}$.

2. Combinatorial maps and a Parity Theorem

To make our objects less dependent of topology we use a combinatorial counterpart for topological maps introduced (with a small variation) in [7,8]. A *combinatorial map* or simply a *map* M is an ordered triple (C_M, v_M, f_M) where: (i) C_M is a connected finite cubic graph; (ii) v_M and f_M are disjoint perfect matchings in C_M, such that each component of the subgraph of C_M induced by $v_M \cup f_M$ is a polygon with 4 edges and it is called an *M-rectangle*.

From the above definition, it follows that C_M may contain double edges but not loops. A third perfect matching in C_M is $E(C_M) - (v_M \cup f_M)$ and is denoted by a_M. The set of diagonals of the M-rectangles, denoted by z_M, is a perfect matching in the complement of C_M. The edges in v_M, f_M, z_M, a_M are called respectively v_M-edges, f_M-edges, z_M-edges, a_M-edges. The graph $C_M \cup z_M$ is denoted by Q_M, and is a regular graph of valence 4. A component induced by $a_M \cup v_M$ is a polygon with an even number of vertices and it is called a v-gon. Similarly, we define an f-gon, and a z-gon, by replacing v for f and v for z. Clearly, the f-gons and z-gons of C_M correspond to the facial paths and the zigzags of M^t. To avoid the use of colors the M-rectangles are presented in the pictures as rectangles in which the short sides (s) are v_M-edges, the long sides (ℓ) are f_M-edges and the diagonals (d) are z_M edges. An *M-rectangle with diagonals* or simply an *M-rectangle* (being understood that the diagonals are present) is a component induced by v_M, f_M, z_M. If π is a permutation of the symbols $s\ell d$, then $M(\pi)$ denotes the map obtained from M by permuting the short sides, the long sides and the diagonals according to π in all $r \in R$. The *dual map* of M is the map $D = M(\ell s d)$; D and M have the same z-gons and the v-gons and f-gons interchanged. The *phial map* of M is the map $P = M(d\ell s)$; P and M have the same f-gons and the v-gons and z-gons interchanged. The *antimap* of M is the map $M^{\sim} = M(sd\ell)$; M and $M^{\sim}$ have the same v-gons and the f-gons and z-gons interchanged. The pairs (M, D), (M, P), $(M, M^{\sim})$ constitute the *map dualities* introduced in [8]. The dual of P is $D^{\sim}$ and the dual of $M^{\sim}$ is $P^{\sim}$ (see Fig. 1).

Given a map M and its dual D, there exists a closed surface, denoted by $\mathrm{Surf}(M, D)$ where $C_M = C_D$ naturally embeds. Consider the v-gons, the f-gons and the M-rectangles bounding disjoint closed disks. Each edge of C_M occurs twice in the boundary of this collection of disks. Identify the collection of disks along the two occurrences of each edge. The result is a closed surface and C_M is *faithfully embedded on it*, meaning that the boundaries of the faces are *bi*colored poly *gons* or *bigons*. Similarly, there are surfaces $\mathrm{Surf}(D^{\sim}, P)$ and $\mathrm{Surf}(P^{\sim}, M^{\sim})$.

We define a function ψ which turns out to be a bijection from the set of maps onto the set of t-maps. We denote $\psi(M)$ by M^t. Given a map M, to obtain M^t we proceed as follows. Consider the t-map (C_M, S), where $S = \mathrm{Surf}(M, D)$, given by the faithful embedding of M. The v-gons, the f-gons and the M-rectangles are boundaries of (closed, in this case) disks embedded (and forming) the surface $S(M)$. Shrink to distinct points the disjoint closed disks bounded by v-gons. The M-rectangles, then, become bounding digons. Shrink each such bounding digon to a line, maintaining its vertices unaffected. With these contractions, effected in S, t-map (C_M, S) becomes, by definition, $M^t = (G_M, S)$. Graph G_M is called the *graph induced* by M. A combinatorial description of G_M can be given as follows: the vertices of G_M are the v-gons of M; its edges are the rectangles of M; the two ends of an edge of G_M are the two v-gons (which may coincide and the edge is a *loop*) that contain the v_M-edges of the corresponding M-rectangle. It is evident that ψ is inversible: given a t-map we replace each edge by a bounding digon in its surface, and then expand each vertex to a disc in order to obtain a cellular embedding of a cubic graph. Therefore, ψ^{-1} is well defined; in fact, it is the dual of a useful construction in topology, namely, barycentric division. Thus, ψ is a bijection from the set of maps onto the set of t-maps. It can be observed that ψ induces a bijection from the set of M-rectangles onto the set of edges of G_M. We use this bijection to identify the sets R and $E(G_M)$. Via the set of M-rectangles (with their diagonals), which is invariant for $\{M, D, P, M^\sim, D^\sim, P^\sim\}$, we identify $E(G_M)$ and $E(G_{M'})$ for $M' \in \{D, P, M^\sim, D^\sim, P^\sim\}$. Denote these identified sets of edges by E, with $|E| = n$.

Consider the function ψ_c^M from the cycle space of C_M, onto the cycle space of G_M. It is defined as follows: for $X \in CS(C_M)$, an edge $e \in E$ is in $\psi_c^M(X)$ if the intersection of the rectangle corresponding to e with X contains exactly one f_M-edge. With this definition, it follows that $\psi_c^M(X)$ is a cycle in G_M and that ψ_c^M is surjective.

Proposition 1. *(See [7].) ψ_c^M is a homomorphism. Its kernel is the subspace of $CS(C_M)$ generated by the v-gons and the rectangles of M.*

Since every element of the kernel of ψ_c^M has an even number of edges of C_M, it follows that if $\psi_c^M(S_1) = \psi_c^M(S_2)$, then $|S_1| \equiv |S_2| \bmod 2$. This observation makes the following definition meaningful. A cycle S in G_M is called an *r-cycle* in M^t if $\psi_c^M(S') = S$ and $|S'|$ is odd, for some cycle S' in C_M. If $|S'|$ is even and $\psi_c^M(S') = S$, then we say that S is an *s-cycle* in M^t. *r-Circuits* are minimal r-cycles. We observe that the r-circuits in M^t are precisely the orientation-reversing polygons in M^t. This topological notion is not used; we work with our parity definition of r-cycle. The following proposition shows that the type of $c_P(i)$ depends only on the parity of $\lambda_{\overline{P}}(i)$.

Proposition 2. *(Parity Theorem [7].) If M is a map with a single z-gon, then $c_P(i)$ is an s-cycle in M^t if and only if $|\lambda_{\overline{P}}(i)|$ is even.*

In the next section we display an example on the Klein bottle. The basic motivation is to exemplify a pair of linear transformations which in general arise from a map M with a single z-gon, so that its phial P has a single vertex. This pair of linear transformations constitutes the main tool to get the main theorems, Propositions 4 and 5.

3. The linear transformations c_P and $c_{P^\sim}$

Consider the example of a t-map M on the Klein bottle given in Fig. 2. It has four vertices eight edges, four faces and a single zigzag given on the left of Fig. 1. The cyclic sequence of edges visited in the zigzag is

$$P = (1, 4, 5, 6, 5, 4, 3, 8, 7, 3, 2, -1, -2, 8, 7, -6).$$

This can be followed in the lacet P'. Note that P' is the medial map of M. The direction of the first occurrence of an edge of G_M defines the orientation. Edges 3, 4, 5, 7, 8 are traversed twice in the positive direction (they correspond to black circles in the medial map) and edges 1, 2, 6 are traversed once in the positive direction and once in the negative direction (they correspond to

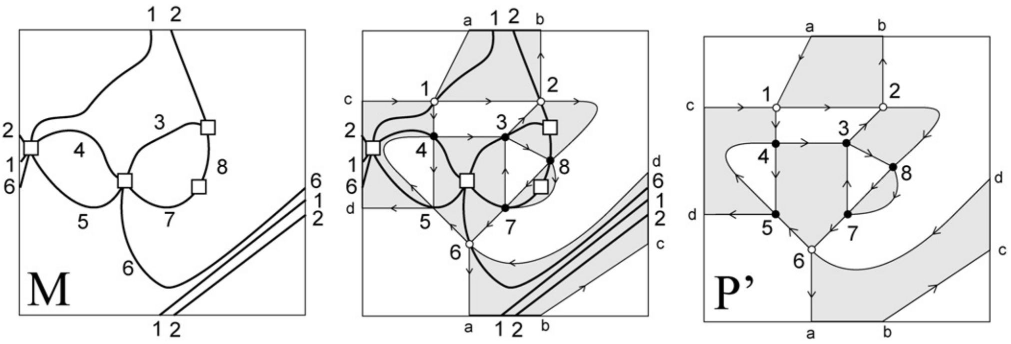

Fig. 2. A t-map in the Klein bottle and its medial P', $|V(P)=1|$. Gauss code $\overline{P}=1456543873212876$.

white circles). The reason for the notation P is that the signed cyclic sequence defines the phial map P (whence also M, D and $P^{\sim}$, as well as the surface of M) and vice-versa, the phial defines the sequence. For the algebraic concepts we refer to [5]. For the graph terminology refer to [1,6]. For more background on graphs embedded into surfaces we refer to [4]. Given a map M with a single z-gon and rectangle set E, we define linear functions (over the field $GF(2)$) $\lambda_{\overline{P}}: 2^E \to 2^E$ and $\tau_P: 2^E \to 2^E$ as follows. They are defined in the singletons and extended by linearity. Let $\lambda_{\overline{P}}(\{i\})$ be the set of edges occurring once in the cyclic sequence P between the two occurrences of edge i. Let $\tau_P(i)=\{i\}$ if i is traversed twice in the same direction in the zigzag path (i is a black vertex in the medial map P'), and $\tau_P(i)=\emptyset$, if i is traversed in opposite direction in the zigzag path (i is a white vertex in P'). If we define t_i as 0 or 1 according to i being white or black in P', then $\tau_P(i)=t_i\{i\}$. We do not distinguish between $A\subseteq E$ and its 0, 1 characteristic vector. Thus the empty set and the zero vector are the same. It is easy to show that given $\overline{P}$ any 0–1 choice for $(t_1, t_2, \ldots, t_{|E|})$ defines P', whence a lacet for $\overline{P}$ in some surface.

Let $c_P=\tau_P+\lambda_{\overline{P}}$. Observe that $c_P(i)$ is the set of edges occurring once in a closed path in G_M. Therefore, $c_P(i)\in CS(G_M)$ and likewise $c_{P^{\sim}(i)}\in CS(G_D)$. In Fig. 2 we see that $c_P(1)=\emptyset\cup\{2,8,7,6\}$, $c_P(3)=\{3\}\cup\{8,7\}$ and, indeed, $\{2,8,7,6\}$ and $\{3,8,7\}$ are in $CS(G_M)$. From the definitions, it follows that if P has a single vertex, for any i, $\tau_P(i)+\tau_{P^{\sim}}(i)=\{i\}$ and that $c_{P^{\sim}}(i)+c_P(i)=\{i\}$ and so, $c_{P^{\sim}}+c_P$ is the identity linear transformation.

Define $b_P=c_{P^{\sim}}\circ c_P=c_P\circ c_{P^{\sim}}=c_P^2+c_P$. The image of b_P is $CS(G_M)\cap CS(G_D)$. Moreover its dimension is 2 minus the Euler characteristic of $\mathrm{Surf}(M,D)$. These facts were first proved in [7]. They also appear in [9], and in [3].

Let $\Lambda(\overline{P})=\Lambda$ be the $n\times n$ matrix with entries in $GF(2)$ where its (i,j)-entry, λ_{ij}, is 1 if and only if $j\in\lambda(i)$ or equivalently, $i\in\lambda(j)$. Denote by K the rectangle of Λ, arithmetic modulo 2. The (i,j)-entry of K is denoted κ_{ij}. If $A\subseteq E$, then let $s_i(A)=\{j\in A \mid t_j=t_i\}$.

Proposition 3 *(Lemma on b_P). The linear transformation $b_P=c_P\circ c_{P^{\sim}}$ satisfies*

$$b_P(i)=\lambda^2_{\overline{P}}(i)+s_i\big(\lambda_{\overline{P}}(i)\big),$$

$$j\in b_P(i) \quad\Leftrightarrow\quad i\in b_P(j) \quad\Leftrightarrow\quad \big|c_P(i)\cap c_{P^{\sim}}(j)\big|\equiv 1 \mod 2$$
$$\Leftrightarrow\quad \kappa_{ij}+(1+t_i+t_j)\lambda_{ij}\equiv 1 \mod 2.$$

Proof. Since $c_P+c_{P^{\sim}}$ is the identity, $b_P=c_P^2+c_P$. Thus,

$$b_P(i) = \left[c_P^2 + c_P\right](i) = \left[(\tau_P + \lambda_{\overline{P}})^2 + (\tau_P + \lambda_{\overline{P}})\right](i)$$
$$= \left[\tau_P^2 + \tau_P\lambda_{\overline{P}} + \lambda_{\overline{P}}\tau_P + \lambda_{\overline{P}}^2 + \tau_P + \lambda_{\overline{P}}\right](i).$$

But $\tau_P^2 = \tau_P$ and so $b_P(i) = [\lambda_{\overline{P}}^2 + \tau_P\lambda_{\overline{P}} + \lambda_{\overline{P}}\tau_P + \lambda_{\overline{P}}](i)$. If i is black in P', then $\lambda_{\overline{P}}\tau_P(i) = \lambda_{\overline{P}}(i)$, and $b_P(i) = [\lambda_{\overline{P}}^2 + \tau_P\lambda_{\overline{P}}](i) = \lambda_{\overline{P}}^2(i) + \tau_P\lambda_{\overline{P}}(i) = \lambda_{\overline{P}}^2(i) + s_i(\lambda_{\overline{P}}(i))$. Note that $\tau_P\lambda_{\overline{P}}(i)$ is the subset of black vertices in $\lambda_{\overline{P}}(i)$. If i is white in P', then $\lambda_{\overline{P}}\tau_P(i) = \emptyset$, and $b_P(i) = [\lambda_{\overline{P}}^2 + \tau_P\lambda_{\overline{P}} + \lambda_{\overline{P}}](i) = \lambda_{\overline{P}}^2(i) + [\tau_P\lambda_{\overline{P}} + \lambda_{\overline{P}}](i) = \lambda_{\overline{P}}^2(i) + s_i(\lambda_{\overline{P}}(i))$. Note that $[\tau_P\lambda_{\overline{P}} + \lambda_{\overline{P}}](i)$ is the subset of white vertices in $\lambda_{\overline{P}}(i)$. This proves the first part.

The equivalences

$$j \in b_P(i) \quad \Leftrightarrow \quad i \in b_P(j) \quad \Leftrightarrow \quad \left|c_P(i) \cap c_{P^\sim}(j)\right| \equiv 1 \mod 2$$

are straightforward from the symmetry of $\lambda_{\overline{P}}$ inducing similar symmetry on b_P and from its definition as $c_P \circ c_{P^\sim}$. Finally we prove $j \in b_P(i) \Leftrightarrow \kappa_{ij} + (1 + t_i + t_j)\lambda_{ij} \equiv 1 \bmod 2$, using the first part of the lemma. We have $j \in b_P(i) \Leftrightarrow [j \in \lambda_{\overline{P}}^2(i)$ and $j \notin s_i(\lambda_{\overline{P}}(i))]$ or $[j \notin \lambda_{\overline{P}}^2(i)$ and $j \in s_i(\lambda_{\overline{P}}(i))] \Leftrightarrow [\kappa_{ij} = 1$ and $(\lambda_{ij} = 0$ or $t_i + t_j = 1)]$ or $[\kappa_{ij} = 0$ and $\lambda_{ij} = 1$ and $t_i + t_j = 0]$ $\Leftrightarrow \kappa_{ij} + (1 + t_i + t_j)\lambda_{ij} \equiv 1 \bmod 2$, establishing the lemma. □

4. Solution of the Gauss code problem in arbitrary surfaces

The *genus of a non-orientable surface* is 2 minus its Euler characteristic, while the *genus of an orientable surface* is the previous value divided by 2. From the parity theorem and the fact that c_P is surjective follow that $\overline{P}$ has all its lacets in an orientable surface if and only if $|\lambda(i)|$ is even for every i. Therefore, if there exists any i with $|\lambda(i)|$ odd, then all the lacets for $\overline{P}$ are in non-orientable surfaces. These facts permit us to classify Gauss codes $\overline{P}$ as *orientable* or *non-orientable* in the obvious way, via the parity of the $|\lambda_{\overline{P}}(i)|$'s, $i = 1, \ldots, n$.

A fundamental observation is that $|c_P(i) \cap c_{P^\sim}(j)| \equiv 1 \bmod 2$ is the intersection number mod 2 of the homology classes of $c_P(i)$ and $c_{P^\sim}(j)$ (they are in dual cellular decompositions and only dual edges intersect). So, this number is invariant (see [15, Sections 66–74]) if we replace $c_P(i)$ and $c_{P^\sim}(j)$ by any homologous cycles. Let $(\xi_i, \xi_2, \ldots, \xi_g, \eta_1, \eta_2, \ldots, \eta_g)$ be the standard homology basis (see Fig. 3) of $\mathbb{T}_2^g$, the orientable surface of genus g, formed by attaching g (orientable) handles to the 2-sphere. Let $(\xi_i^\star, \xi_2^\star, \ldots, \xi_g^\star, \eta_1^\star, \eta_2^\star, \ldots, \eta_g^\star)$ be the standard

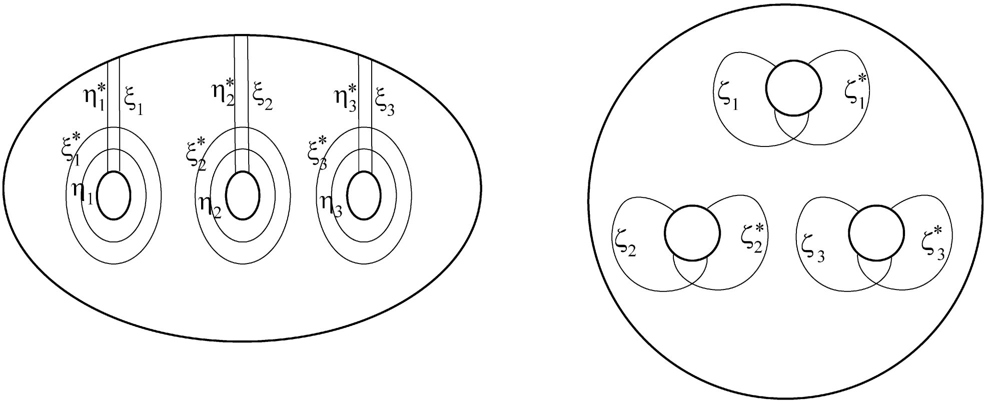

Fig. 3. Dual standard basis for the homology of $\mathbb{T}_g^2$ and $\mathbb{P}_g^2$ (example for the case $g = 3$).

homology dual basis (see also Fig. 3) of $\mathbb{T}_2^g$. There are unique x_{ih}, y_{ih}, $x^\star_{jh}$, $y^\star_{jh}$ in $GF(2)$ such that

$$c_P(i) \sim \sum_{h=1}^{g} x_{ih}\xi_h + \sum_{h=1}^{g} y_{ih}\eta_h, \qquad c_{P^\sim}(j) \sim \sum_{h=1}^{g} x^\star_{jh}\xi^\star_h + \sum_{h=1}^{g} y^\star_{ih}\eta^\star_h.$$

The intersection number mod 2 of the homology classes of $c_P(i)$ and $c_{P^\sim}(j)$ can be expressed as $\sum_{h=1}^{g}(x_{ih}x^\star_{jh} + y_{ih}y^\star_{jh})$ mod 2, since ξ_h only intersects its dual $\xi^\star_h$ and η_h only intersects its dual $\eta^\star_h$. Note that the elements of the standard dual pair of basis satisfy $\xi^\star_h \sim \eta_h$ and $\eta^\star_h \sim \xi_h$, $h = 1, \ldots, g$. Therefore,

$$c_{P^\sim}(j) \sim \sum_{h=1}^{g} x^\star_{jh}\xi^\star_h + \sum_{h=1}^{g} y^\star_{ih}\eta^\star_h \sim \sum_{h=1}^{g} x^\star_{jh}\eta_h + \sum_{h=1}^{g} y^\star_{ih}\xi_h.$$

Now consider

$$c_P(j) \sim \sum_{h=1}^{g} x_{jh}\xi_h + \sum_{h=1}^{g} y_{jh}\eta_h,$$

and take into account that $c_P(j) \sim c_{P^\sim}(j)$: both of these cycles are homologous to a common pre-image under Ψ_c^M and Ψ_c^D. It follows that $x^\star_{jh} = y_{jh}$ and $y^\star_{jh} = x_{jh}$. We then get

$$\left|c_P(i) \cap c_{P^\sim}(j)\right| \equiv \sum_{h=1}^{g}(x_{ih}y_{jh} + y_{ih}x_{jh}) \mod 2. \tag{1}$$

We turn now to the non-orientable surface of genus g, denoted by $\mathbb{P}_g^2$, formed by attaching g cross-caps to the 2-sphere. It has standard basis $(\zeta_1, \zeta_2, \ldots, \zeta_g)$ and dual standard basis $(\zeta^\star_1, \zeta^\star_2, \ldots, \zeta^\star_g)$, see Fig. 3. We have unique z_{ih} and $z^\star_{jh}$ satisfying

$$c_P(i) \sim \sum_{h=1}^{g} z_{ih}\zeta_h, \qquad c_{P^\sim}(j) \sim \sum_{h=1}^{g} z^\star_{jh}\zeta^\star_h.$$

A similar, but simpler reasoning shows that $\zeta_h \sim \zeta^\star_h$ and $z_{jh} = z^\star_{jh}$ implying

$$\left|c_P(i) \cap c_{P^\sim}(j)\right| \equiv \sum_{h=1}^{g} z_{ih}z^\star_{jh} \equiv \sum_{h=1}^{g} z_{ih}z_{jh} \mod 2. \tag{2}$$

By a *tight* solution to the system in the next proposition we mean that for each h there is an i so that $x_{ih} = 1$ or $y_{ih} = 1$. If a solution is not tight there will be unused handles which can be discarded (g can be made smaller).

Proposition 4 *(Theorem on orientable Gauss codes). Let $\overline{P}$ be an orientable Gauss code. Then the set of lacets for $\overline{P}$ in a orientable surface of genus g are in* 1–1 *correspondence with the tight solutions of the quadratic system of n^2 equations*

$$\kappa_{ij} + (1 + t_i + t_j)\lambda_{ij} = \sum_{h=1}^{g}(x_{ih}y_{jh} + x_{jh}y_{ih}) \quad \forall (i, j), \tag{3}$$

where the unknowns are t_i, x_{ih} and y_{ih}, $1 \leqslant i \leqslant n$, $1 \leqslant h \leqslant g$.

For the non-orientable case by a *tight* solution to the system in the next proposition we mean that for each h there is an i so that $z_{ih} = 1$. If a solution is not tight there will be unused cross-caps which can be discarded (g can be made smaller).

Proposition 5 *(Theorem on non-orientable Gauss codes). Let $\overline{P}$ be a non-orientable Gauss code. Then the set of lacets for $\overline{P}$ in a non-orientable surface of genus g are in 1–1 correspondence with the tight solutions of the quadratic system of n^2 equations*

$$\kappa_{ij} + (1 + t_i + t_j)\lambda_{ij} = \sum_{h=1}^{g} z_{ih} z_{jh} \quad \forall (i, j), \tag{4}$$

where the unknowns are t_i and z_{ih}, $1 \leqslant i \leqslant n$, $1 \leqslant h \leqslant g$.

Proof. The theorems are proved together, as a consequence of the second part of Proposition 3 and the above orientable and non-orientable formulas (1) and (2) for $|c_P(i) \cap c_{P^\sim}(j)|$. Any choice of $t = (t_1, t_2, \ldots, t_n)$ produces a lacet for $\overline{P}$ in some surface. The existence of solutions in genus g is a necessary and sufficient condition for the embedding to be on the surface of genus g. Of course, we are interested in the smallest g that produces a consistent system. □

5. Linear systems

The necessary and sufficient condition for the existence of lacets for $\overline{P}$ on the 2-sphere is the case $g = 0$ of system (3):

$$(1 + t_i + t_j)\lambda_{ij} = \kappa_{ij} \quad \forall (i, j). \tag{5}$$

This is a linear system on the variables t_i. As we show below the analysis of its inconsistency implies the characterization given by Rosenstiehl [13,14].

We have a similar reduction from quadratic to linear equations for the non-orientable cases of genus 1 (projective plane) and 2 (Klein bottle). Consider the equation of system (4) corresponding to (i, i). Since $\lambda_{ii} = 0$ and $z_{ih}^2 = z_{ih}$, we have

$$\kappa_{ii} + (1 + t_i + t_i)\lambda_{ii} = \sum_{h=1}^{g} z_{ih} z_{ih} \quad \Leftrightarrow \quad \kappa_{ii} = \sum_{h=1}^{g} z_{ih}.$$

Suppose $g = 1$. Then $z_{i1} = \kappa_{ii}$ and $z_{j1} = \kappa_{jj}$, therefore the system becomes

$$(1 + t_i + t_j)\lambda_{ij} = \kappa_{ij} + \kappa_{ii}\kappa_{jj} \quad \forall (i, j). \tag{6}$$

Note that the variables z_{ih} disappear and we get an equivalent linear system on the variables t_i. As we show next the characterization given in Lins' thesis [7] follows from analyzing the inconsistency of this linear system.

If $\overline{P}$ is orientable, let $\kappa'_{ij} = \kappa_{ij}$. If it is non-orientable, let $\kappa'_{ij} = \kappa_{ij} + \kappa_{ii}\kappa_{jj}$. The two systems above become a single one

$$(1 + t_i + t_j)\lambda_{ij} = \kappa'_{ij} \quad \forall (i, j). \tag{7}$$

A *bad edge* is a pair (i, j) so that $\lambda_{ij} = 0$ and $\kappa'_{ij} = 1$. By abuse of language, denote also by Λ the graph whose adjacency matrix is Λ. A *bad polygon* in Λ is a polygon in it so that its number of edges (i, j) with $\kappa'_{ij} = 0$ is odd. We have the following corollary.

Proposition 6 *(Corollary for $g \leqslant 1$). Let $\overline{P}$ be a Gauss code. It has a solution in the 2-sphere or in the projective plane if and only if it does not have a bad edge nor a bad polygon.*

Proof. If the system (7) is consistent it clearly does not contain a bad edge. If it contains a bad polygon summing the equations corresponding to the edges in the polygon produces the inconsistency $1 = 0$. Conversely, if the system is inconsistent there exists a minimal subset of equations whose sum produces the inconsistency $1 = 0$. From the simplicity of the equations in the system, it follows easily that this set of minimal equations corresponds to a single one (a bad edge) or to a subset of equations defining a bad polygon. □

This result is equivalent to the characterization of Rosenstiehl [13,14] and of Lins [7].

Now suppose $g = 2$. Then $\sum_{h=1}^{2} z_{ih} z_{jh} = z_{i1} z_{j1} + z_{i2} z_{j2} = z_{i1} z_{j1} + (\kappa_{ii} + z_{i1})(\kappa_{jj} + z_{j1}) = z_{i1} z_{j1} + \kappa_{ii}\kappa_{jj} + \kappa_{ii} z_{j1} + z_{i1}\kappa_{jj} + z_{i1} z_{j1}$. Therefore, $\sum_{h=1}^{2} z_{ih} z_{jh} = \kappa_{ii}\kappa_{jj} + \kappa_{ii} z_{j1} + \kappa_{jj} z_{i1}$. The necessary and sufficient condition for the solution in the Klein bottle becomes existence of solutions for

$$\kappa_{ij} + (1 + t_i + t_j)\lambda_{ij} = \kappa_{ii}\kappa_{jj} + \kappa_{ii} z_{j1} + \kappa_{jj} z_{i1} \quad \forall (i, j).$$

The variables z_{i2}, $i = 1, \ldots, n$, disappear and the system becomes linear. However we do not have (as in the case $g \leqslant 1$) a simple combinatorial description in graph theoretical structures of a minimal set of inconsistent equations. From the algorithmic point of view, this is irrelevant, since we can display the inconsistency.

6. A final remark and acknowledgments

A basic open problem in this theory is to find an algorithmic characterization of the orientable Gauss codes which do not embed in the torus. We thank Bruce Richter for bringing the Crapo–Rosenstiehl paper to our attention. We also thank an anonymous referee who read carefully a previous version of this work giving valuable suggestions for its improvement. The first author acknowledges the partial support of CNPq (Process number 306106/2006). The second author acknowledges the partial support for FAPESP contract 2006/03360-4.

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